GOVERNMENT OF TAMILNADU

DIRECTORATE OF TECHNICAL EDUCATION

CHENNAI – 600 025

STATE PROJECT COORDINATION UNIT

Diploma in Instrumentation and Control Engineering

Course Code: 1042

M – Scheme

e-TEXTBOOK on

CONTROL ENGINEERING

For 5th Semester DICE

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34253- CONTROL ENGINEERING

DETAILED SYLLABUS

UNIT	NAME OF THE TOPIC	HOURS
I	**BASICS OF CONTROL SYSTEMS, LAPLACE TRANSFORM AND TRANSFER FUNCTION** System – Linear & Non Linear, Continuous & Discrete - Control system - open loop & closed loop –Examples – basics of Laplace transform – Inverse Laplace transform – Transfer function –order and type of a transfer function – pole/ zero plot - Transfer function of Translational Mechanical system (simple second order system with one mass) – Transfer function of Electrical systems using R,L,C	**17 HRS**
II	**BLOCK DIAGRAM AND SIGNAL FLOW GRAPH REPRESENTATION** Block diagram: Introduction – advantages – rules for block diagram reduction – simple problems. Signal flow graph: Rules for reduction – Mason's gain formula – applications of Mason's formula – simple problems – comparison of block diagram reduction and signal flow graph methods.	**18 HRS**
III	**TIME RESPONSE** Standard test signals (step, ramp, sine and Parabolic) – order and Type of system - I order, II order system – derivation – step response of I order, II order system – time domain specifications (definition & formulas only) – steady state error, static error constants – problems.	**16 HRS**
IV	**FREQUENCY RESPONSE** Frequency response of linear system –Advantages – Frequency domain specifications (definitions only) – bode plot – gain margin – phase margin – problems – polar plot – problems.	**16 HRS**
V	**STABILITY** Definition –Location of the roots on the s-plane for stability absolute stability – relative stability– characteristic equation – Routh's stability criterion technique – construction of root locus – problems.	**16 HRS**
	Revision and Test	**7 Hrs**

TEXT BOOKS:

1) Control systems by A.Nagoorkani, RBA publishers,2006(Page no. 1-36, 70- 129, 255-280, 284-327, 343-417, 455- 490)

REFERENCE BOOKS

1. Automatic control system by Benjamin S.Kuo,Printice Hall of India Pvt. Ltd., Seventh edition,1995.
2. Advanced control theory by I.J.Nagrath and M.Gopal, New Age international publishers, II edition, 2002
3. Control systems by A. Anandkumar, EEE, PHI
4. Control Engineering Theory & Practice by M.N. Bandyopadhiyay, PHI

UNIT –I

BASICS OF CONTROL SYSTEMS, LAPLACE TRANSFORM AND TRANSFER FUNCTION

1.1 SYSTEM:

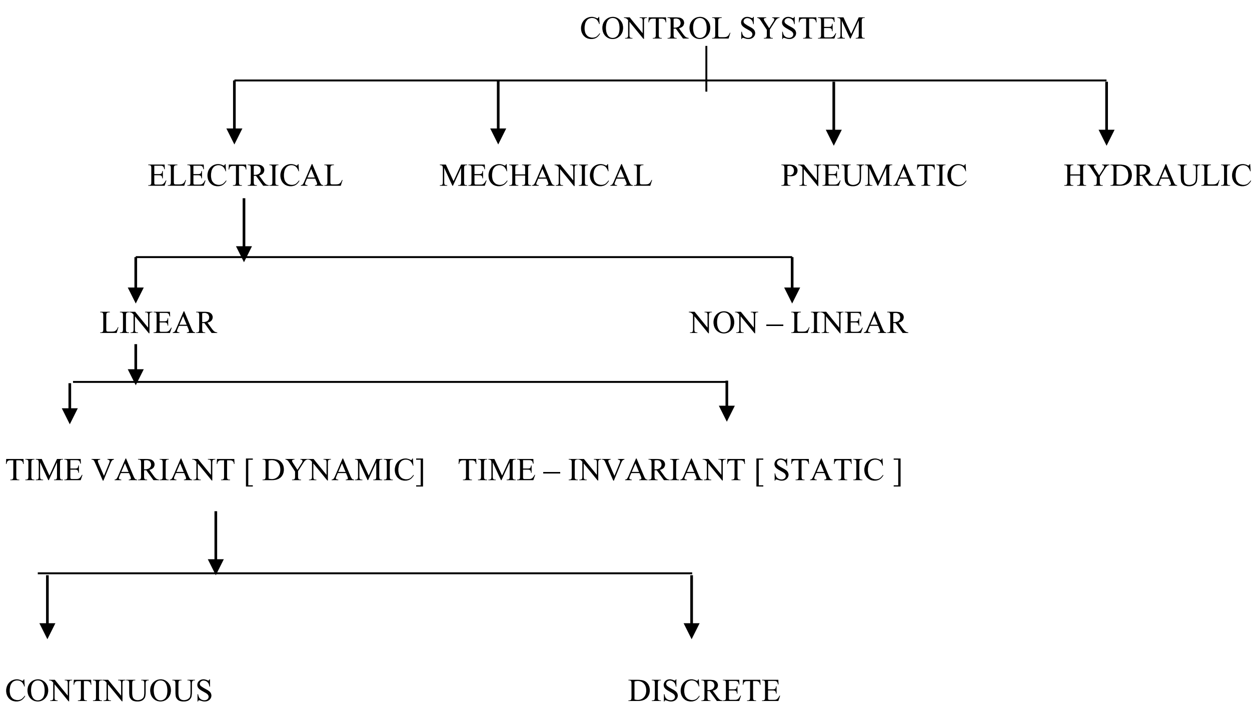
A number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.

1.2 CONTROL SYSTEM:

In a system when the output quantity is controlled by varying the input quantity then the system is called control system. The output quantity is called controlled variable or response and the input quantity is called command signal or excitation.

1.3 CLASSIFICATION OF CONTROL SYSTEM:

1.Open loop and closed loop system.
2.SISO and MIMO.
3.According to the type of component control system can be classified as Electrical, Hydraulic, Pneumatic, Mechanical etc.,



1.3.1 LINEAR & NON-LINEAR SYSTEM:

LINEAR SYSTEM:

A system is said to be linear if it obeys the principle of superposition and homogeneity. The principle of superposition states that the response of the system to a weighted sum of the responses of the system to each individual input signals.

The system is said to be linear, if it satisfies the following two properties:

Adaptive property that is for any x and y belonging to the domain of the function f, we have

$$\boxed{F(x + y) = f(x) + f(y)}$$

Homogeneous property that is for any x belonging to the domain of the function f and for any scalar constant α, we have

$$\boxed{F(\alpha x) = \alpha f(x)}$$

Homogeneity: A system is said to be homogeneous, if we multiply input with some constant 'A' then output will also be multiplied by the same value of constant.

NON-LINEAR SYSTEM:

A system which does not obey the principle of Superposition theorem then it is called as Non-linear system.

1.3.2 CONTINUOUS (OR) ANALOG SYSTEM:

In these types of control system, we have continuous signal as the input to the system. These signals are the continuous function of time. We may have various sources of continuous input signal like sinusoidal type signal input source, square type of signal input source, signal may be in the form of continuous triangle etc.

1.3.3 DISCRETE (OR) DIGITAL SYSTEM:

In these types of control system, we have discrete signal (or signal may be in the form of pulse) as the input to the system. These signals have the discrete interval of time. We can convert various sources of continuous input signal like sinusoidal type signal input source, square type of signal input source etc into discrete form using the switch.

1.3.4 OPEN LOOP SYSTEM:

Any physical system which does not automatically correct the variation in its output is called an open loop system or control system, in which the output quantity has no effect upon the input quantity are called open-loop control system. This means that the output is not fedback to the input for correction.

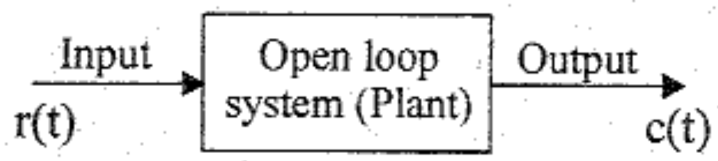


Fig.1.1 Open Loop System

EXAMPLE OF OPEN LOOP SYSTEM (TEMPERATURE CONTROL SYSTEM):

The electric furnace is shown in fig 1.2. It is an open loop system. The output in the system is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to heater remains ON.

The ON and OFF of the supply is governed by the time setting of the relay. The temperature Is measured by a sensor which gives an analog voltage corresponding to the temperature of the furnace. The analog signal is converted to digital signal by an Analog - to - Digital converter (A/D converter).

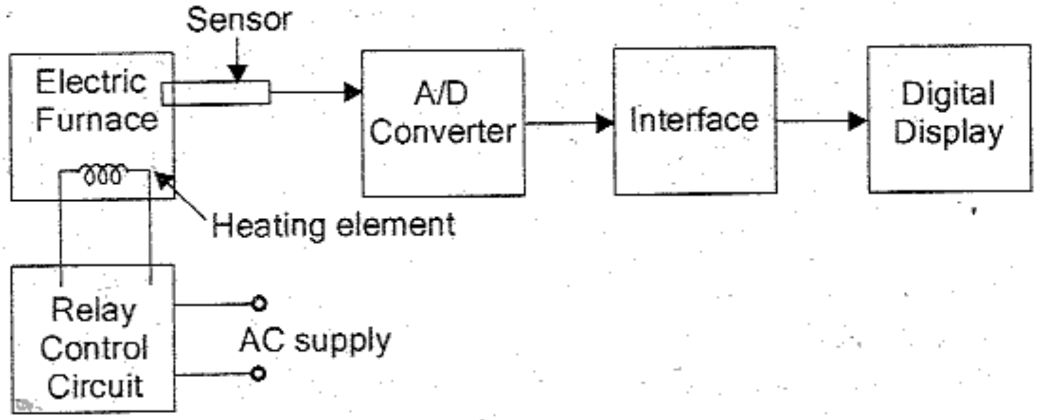


Fig.1.2 Example of Open Loop System

The digital signal is given to the digital display device to display the temperature: In this system If there is any change in output temperature then the time setting of the relay is not altered automatically.

ADVANTAGES OF OPEN LOOP SYSTEMS:

- The open loop systems are simple and economical.
- The open loop systems are easier to construct.
- Generally, the open loop systems are stable.

DISADVANTAGES OF OPEN LOOP SYSTEMS:

- The open loop systems are inaccurate and unreliable.
- The changes in the output due to external disturbances are not corrected automatically.

1.3.5 CLOSED LOOP SYSTEM:

Control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called closed loop systems. The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called an automatic control system.

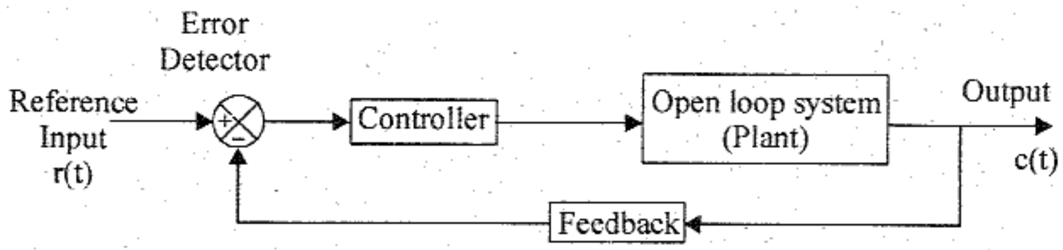


Fig.1.3 Closed Loop System

It consists of an error detector, a controller, plant (open loop system) and feedback path elements. The reference signal (or) input signal corresponds to the desired output. The feedback path elements sample the output and converts it to a signal of same type as that of reference signal. The feedback signal is proportional to output signal and it is fed to the error detector. The error signal generated by the error detector is the difference between reference signal and feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

EXAMPLE OF CLOSED LOOP SYSTEM (TEMPERATURE CONTROL SYSTEM:

A Closed loop system of the electric furnace shown in fig. The output of the system is the desired temperature and it depends on the time during which the supply to healer remains ON. The switching ON and OFF of the relay is controlled by a control switch is a digital system or computer. The desired temperature is input to the system through keyboard or as a signal corresponding to the desired temperature via ports.

The actual temperature is sensed by sensor and converted to digital signal by the A/D converter. The computer reads the actual temperature and compares with desired temperature. If It finds any difference then It sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus, the system automatically corrects any changes in output Hence it is a closed loop system.

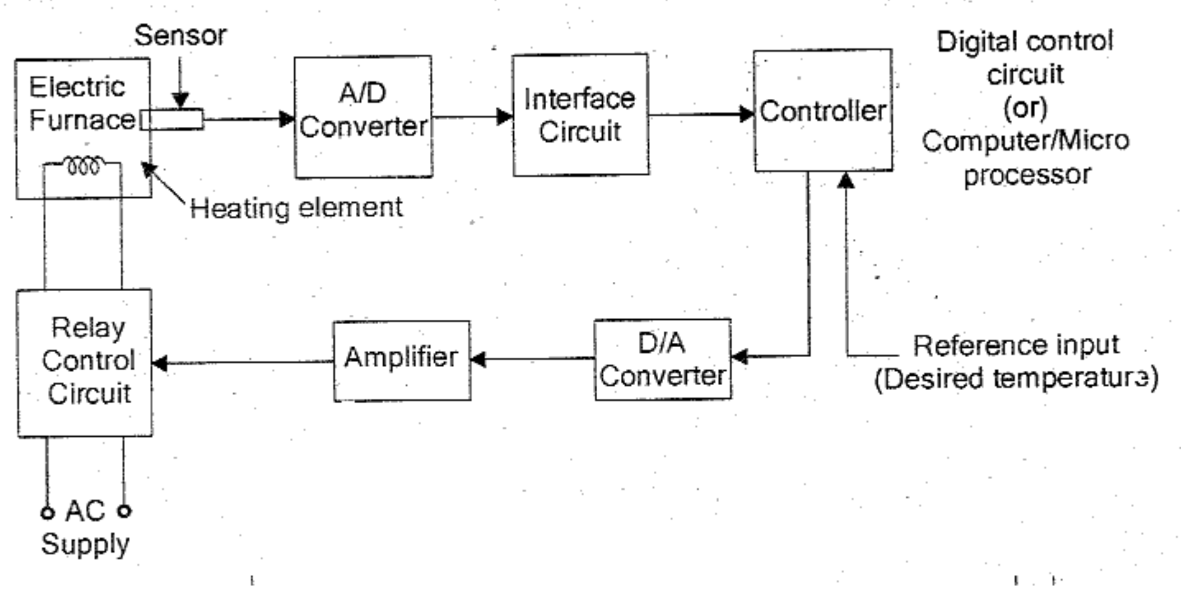


Fig.1.4 Example of Closed Loop System

ADVANTAGES OF CLOSED LOOP SYSTEMS:

- The closed loop systems are accurate.
- The Closed loop systems are accurate even in the presence of non-linearities.
- The sensitivity of the systems may be made small to make the system more stable.
- The closed loop systems are less affected by noise.

DISADVANTAGES OF CLOSED LOOP SYSTEMS:

- The closed loop systems are complex and costly.

- The feedback in closed loop system may lead to oscillatory response.
- The feedback reduces the overall gain of the system.
- Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system.

DIFFERENCE BETWEEN OPEN LOOP AND CLOSED LOOP SYSTEM:

OPEN LOOP SYSTEM	**CLOSED LOOP SYSTEM**
Inaccurate and unreliable	Accurate and reliable
Simple and economical	Complex and costlier
The changes in output due to external disturbances are corrected by manually	The changes in output due to external disturbances are corrected by automatically
Generally stable	Great effects are needed to design stable system
Highly affected by noise	Less affected by noise
No feedback element	Feedback element is present

1.4 LAPLACE TRANSFORM:

The Laplace transform provides a useful method of solving certain types of differential equations when certain initial conditions are given, especially when the initial values are zero. Let f(t) be the function of t, time for all t=0, then the Laplace transform of f(t) can be defined as,

$$\text{Laplace transform of f(t)} = \mathcal{L}\,[f(t)] = F(S) = \int_0^\infty e^{-st} f(t)dt \qquad \text{when } t \geq 0$$

1.4.1 LAPLACE TRANSFORM FORMULAS:

$\mathcal{L}$[f(t)] = F(S)	
f(t)	**F(S)**
1	$\frac{1}{s}$

t^n, n=1,2,3…	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$
Sin at	$\frac{a}{s^2+a^2}$
Cos at	$\frac{s}{s^2+a^2}$
Cosh at	$\frac{a}{s^2-a^2}$
Sinh at	$\frac{s}{s^2-a^2}$
e^{at}cos bt	$\frac{(s-a)}{(s-a)^2+b^2}$
e^{at}sin bt	$\frac{b}{(s-a)^2+b^2}$
e^{at}cosh bt	$\frac{(s-a)}{(s-a)^2-b^2}$
e^{at}sinh bt	$\frac{b}{(s-a)^2-b^2}$
e^{-at}cos bt	$\frac{(s+a)}{(s+a)^2+b^2}$
e^{-at}sin bt	$\frac{b}{(s+a)^2+b^2}$
e^{-at}cosh bt	$\frac{(s+a)}{(s+a)^2-b^2}$
e^{-at}sinh bt	$\frac{b}{(s+a)^2-b^2}$

1.4.2 BASICS OF LAPLACE TRANSFORM:

Property 1: Laplace transform of f(t) = 1

If y = f(t) = 1, then $\mathcal{L}$ [1] = $\int_0^\infty e^{-st} f(t)\, dt$

$\int_0^\infty e^{-st}\ f(t)\ \mathrm{dt} = \left[\frac{e^{-st}}{-s}\right] \quad = \left[\frac{e^{-st}}{-s}\right]_0^\infty$

$\boxed{\mathcal{L}\,(1)\ = \left[\frac{1}{s}\right]}$

Property 2: Laplace transform of f(t) = t^n , When t > 0

Solution:

Given f(t) = t^n

Apply Laplace Transform

$\mathcal{L}$[f(t)] =$\mathcal{L}$[t^n]

=$\int_0^\infty e^{-st}$ f(t) dt = $\int_0^\infty e^{-st}\ t^n$dt

Let ts = x

t = $\frac{X}{s}$

dt = $\frac{dX}{s}$

$\mathcal{L}[\ t^n\ ] = \int_0^\infty e^{-x}\ (\frac{X}{s})^n(\frac{dX}{s})\mathrm{dt}\ = \int_0^\infty e^{-x}\ (\frac{x^n}{s^n})(\frac{dX}{s})$

$= \int_0^\infty e^{-x}\ (\frac{x^n\ dx}{s^{n+1}}) = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}\, x^n dx$

$\int_0^\infty e^{-x}\, x^n dx$ = n! Apply

$\boxed{\mathcal{L}\ [\ t^n\ ] = \frac{\mathrm{n!}}{s^{n+1}}}$

Property 3: Laplace transform of f(t) =e^{at}, When t > 0

Solution:

Given f(t) = $\int_0^\infty \mathrm{e^{-st}f(t)dt} = \int_0^\infty \mathrm{e^{-st}e^{at}dt}$

$= \int_0^\infty \mathrm{e^{-st+at}dt} = \int_0^\infty \mathrm{e^{(-s+a)t}}\ \mathrm{dt}$

$= \int_0^\infty \mathrm{e^{-(s-a)t}}\ \mathrm{dt} = \frac{\mathrm{e^{-(s-a)t}}}{-(\mathrm{s-a})}$

$= -\frac{1}{\mathrm{s-a}}\ \left(\frac{1}{\mathrm{e^{(s-a)\,t}}}\right) = -\frac{1}{\mathrm{s-a}}\ \left(\frac{1}{\mathrm{e^{(s-a)\infty}}} - \frac{1}{\mathrm{e^{(s-a)0}}}\right)$

$= -\frac{1}{\mathrm{s-a}}\left(\frac{1}{\mathrm{e^\infty}} - \frac{1}{\mathrm{e^0}}\right) = -\frac{1}{\mathrm{s-a}}\left(\frac{1}{\infty} - \frac{1}{0}\right)$

$= -\frac{1}{\mathrm{s-a}}\,(0-1) = -\frac{1}{\mathrm{s-a}}\,(-1) = \frac{1}{\mathrm{s-a}}$

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}}$$

Property 4: Laplace transform of f(t) = cos at, When t > 0

Solution:

Given f(t) = cos at

Apply Laplace Transform

$$\mathcal{L}\,(\mathbf{cos\,at}) = L\left[\frac{e^{iat}+e^{-iat}}{2}\right] = \frac{1}{2}\left[e^{iat} + e^{-iat}\right] = \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right]$$

Take LCM.,

$$= \frac{1}{2}\left[\frac{(s+ia)+(s-ia)}{(s-ia)(s+ia)}\right] = \frac{1}{2}\left(\frac{s+\cancel{ia}+s-\cancel{ia}}{s^2+\cancel{sia}-\cancel{sia}-i^2a^2}\right) = \left(\frac{s}{s^2+a^2}\right)$$

$$\boxed{\mathcal{L}\,(\mathbf{cos\,at}) = \left(\frac{s}{s^2+a^2}\right)}$$

Property 5: Laplace transform of f(t) = sinh at when t > 0

Solution:

Given f(t) = **sinh at**

Apply Laplace Transform

$$\mathcal{L}\,(\mathbf{sinh\,at}) = \mathcal{L}\left[\frac{e^{at}-e^{-at}}{2}\right] = \frac{1}{2}\left[e^{at} - e^{-at}\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

Take LCM.,

$$= \frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right] = \frac{1}{2}\left(\frac{(\cancel{s}+a-\cancel{s}+a)}{s^2+\cancel{sa}-\cancel{sa}-a^2}\right) = \left(\frac{a}{s^2-a^2}\right)$$

$$\boxed{\mathcal{L}\,(\sinh at) = \left(\frac{a}{s^2-a^2}\right)}$$

1.4.3 PROBLEMS OF LAPLACE TRANSFORM:

Problem 1:

Find Laplace Transform of f(t) =5+ sinh 6t

Solution:

Given f(t) =5+ sinh 6t

Apply Laplace Transform

$\mathcal{L}$ [f(t)] = $\mathcal{L}$ (5) + $\mathcal{L}$ (sinh 6t)

$$F(s) = \frac{5}{s} + \frac{6}{s^2-6^2} = \frac{5}{s} + \frac{6}{s^2-36}$$

$$\boxed{F(s) = \frac{5}{s} + \frac{6}{s^2-36}}$$

Problem 2:

Find the Laplace Transform of each of the following function:

(a) f(t) = t^2 (b) f(t) = cos 5t

Solution:

(a) f(t) = t^2

Formulae:

$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$, n=1,2,3….

$\mathcal{L}(t^2) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

$$\boxed{\mathcal{L}\left(t^2\right) = \frac{2}{s^3}}$$

(b) f(t) = cos5t

From the table, we find $\mathcal{L}$ (cos at)= $\frac{S}{S^2+a^2}$

Therefore, $\mathcal{L}$ (cos 5t) = $\frac{s}{s^2+5^2} = \frac{s}{s^2+25}$

$$\boxed{\mathcal{L}(\cos 5t) = \frac{s}{s^2+25}}$$

Problem 3:

Find the Laplace Transform of each of the following function:

f(t) = $2t^2 - 4t + 1$

Solution:

$\mathcal{L}$ [f(t)] = 2 $\mathcal{L}$ [t^2] - 4 $\mathcal{L}$ [t] + $\mathcal{L}$ [1]

Apply Laplace formula

F(s) = $2\left(\frac{2!}{s^3}\right) - [4]\left(\frac{1!}{s^2}\right) + \frac{1}{s}$

$$\boxed{F(s) = \frac{4-4s+s^2}{s^3}}$$

Problem 4:

Find the Laplace Transform of each of the following function:

f(t) = 2 sin 3t - e^{2t}

Solution:

$\mathcal{L}$ (2 sin 3t - e^{2t}) = 2 $\mathcal{L}$ (sin 3t) – L (e^{2t})

$$F(s) = 2\left(\frac{3}{S^2+3^2}\right) - \left(\frac{1}{S-2}\right)$$

$$\boxed{F(s) = \left(\frac{6}{S^2+9}\right) - \left(\frac{1}{S-2}\right)}$$

Problem 5:

Find the Laplace Transform of each of the following function:

f(t) = $e^{-2t} \sin 3t$

Solution:

Given f(t) = e^{-2t} sin 3t

Apply Laplace Transform

$\mathcal{L}$ [f(t)] = $\mathcal{L}$ [e^{-2t} sin 3t]

$$\mathcal{L}\,(e^{-at}\sin bt) = \frac{b}{(s+a)^2+b^2} = \frac{3}{(s+2)^2+3^2} = \frac{3}{(s+2)^2+9}$$

$$\boxed{\mathcal{L}\,(e^{-2t}\sin 3t) = \frac{3}{s^2+4s+13}}$$

Problem 6:

Find the Laplace Transform of each of the following function:

f(t) = [$e^{-5t} + e^{-2t} \sinh 3t$]

Solution:

Given f(t) = [e^{-5t} + e^{-2t} sinh 3t]

$\mathcal{L}$ [f(t)] = $\mathcal{L}$ [e^{-5t} + e^{-2t} sinh 9t]

= $\mathcal{L}$ [e^{-5t}] + $\mathcal{L}$ [e^{-2t}sinh9t]

$$= \left(\frac{1}{s+5}\right) + \left(\frac{9}{(s+2)^2-9^2}\right) = \left(\frac{1}{s+5}\right) + \left(\frac{9}{(s+2)^2-81}\right) = \left(\frac{1}{s+5}\right) + \left(\frac{9}{(s^2+4s+4-81)}\right)$$

$$\boxed{F(s) = \left(\frac{1}{s+5}\right) + \left(\frac{9}{(s^2+4s-77)}\right)}$$

Problem 7:

Find the Laplace Transform of each of the following function:

g(t) = [$\cosh \omega t + t^3$]

Solution:

Given $g(t) = [\cosh \omega t + t^3]$

$\mathcal{L}[g(t)] = \mathcal{L}[\cosh \omega t + t^3]$

$$= \mathcal{L}[\cosh \omega t] + \mathcal{L}[t^3] = \frac{s}{(s^2-\omega^2)} + \frac{3!}{s^4}$$

$$\boxed{G(s) = \frac{s}{(s^2-\omega^2)} + \frac{6}{s^4}}$$

Problem 8:

Find the Laplace Transform of each of the following function:

$f(t) = [t^2 e^{-2t}]$

Solution:

Given $f(t) = [t^2 e^{-2t}]$

$\mathcal{L}(e^{-at} f(t)) = F(s+a)$

$f(t) = t^2$; $\mathcal{L}[f(t)] = \mathcal{L}[t^2] = \frac{2}{s^3}$

$$\mathcal{L}[e^{-2t} t^2] = \left(\frac{2}{s^3}\right)_{s \to s+2} = \frac{2}{(s+2)^3}$$

$$\boxed{F(s) = \frac{2}{(s+2)^3}}$$

Problem 9:

Find the Laplace Transform of the following function:

$f(t) = [e^{2t} \cos 5t]$

Solution:

Given $f(t) = [e^{2t} \cos 5t]$

$\mathcal{L}(e^{at} f(t)) = F(s-a)$

$f(t) = \cos 5t$

$$\mathcal{L}[f(t)] = \mathcal{L}[\cos 5t] = \frac{s}{(s^2+5^2)} = \frac{s}{(s^2+25)}$$

$$\mathcal{L}[e^{2t} \cos 5t] = \left(\frac{s}{(s^2+25)}\right)_{s \to s-2}$$

$$\boxed{F(s) = \frac{(s-2)}{(s-2)^2+25}}$$

$$= \frac{1}{2S} + \frac{s}{(s^2+4^2)} = \frac{1}{2S} + \frac{s}{(s^2+16)}$$

$$\boxed{F(s) = \frac{1}{2S} + \frac{s}{(s^2+16)}}$$

Problem 10:

Find the Laplace Transform of each of the following function:

f(t) = $e^{4t}\cos 3t$

Solution:

Given f(t) = = $e^{4t}\cos 3t$

Apply Laplace Transform

$\mathcal{L}$ [f(t)] = $\mathcal{L}$ [= $e^{4t}\cos 3t$]

$\mathcal{L}$ ($e^{4t}\cos at$) = $\frac{(s-a)}{(s-a)^2+b^2}$

$\mathcal{L}$ [$e^{4t}\cos 3t$] = $\frac{(s-4)}{(s-4)^2+3^2} = \frac{(s-4)}{(s-4)^2+9}$

$$\boxed{\mathcal{L}\,[\, e^{4t}\cos 3t] = \frac{(s-4)}{(s^2-4s+25)}}$$

1.5 INVERSE LAPLACE TRANSFORMS:

If $\mathcal{L}\,[f(t)] = F(s)$, then $f(t)$ is called the Inverse Laplace Transform of $F(s)$ and is written as

$$\boxed{\boldsymbol{\mathcal{L}^{-1}[F(s)] = f(t)}}$$

1.5.1 FORMULAS OF LAPLACE INVERSE TRANSFORM:

$\boldsymbol{\mathcal{L}^{-1}[F(s)] = f(t)}$	
$F(s)$	$f(t)$
$\frac{1}{s}$	1
$\frac{n!}{s^{n+1}}$	t^n, n=1,2,3…
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s+a}$	e^{-at}
$\frac{a}{s^2+a^2}$	Sin at

$\frac{s}{s^2+a^2}$	Cos at
$\frac{a}{s^2-a^2}$	Cosh at
$\frac{s}{s^2-a^2}$	Sinh at
$\frac{(s-a)}{(s-a)^2+b^2}$	e^{at} cos bt
$\frac{b}{(s-a)^2+b^2}$	e^{at} sin bt
$\frac{(s-a)}{(s-a)^2-b^2}$	e^{at} cosh bt
$\frac{b}{(s-a)^2-b^2}$	e^{at} sinh bt
$\frac{(s+a)}{(s+a)^2+b^2}$	e^{-at} cos bt
$\frac{b}{(s+a)^2+b^2}$	e^{-at} sin bt
$\frac{(s+a)}{(s+a)^2-b^2}$	e^{-at} cosh bt
$\frac{b}{(s+a)^2-b^2}$	e^{-at} sinh bt

Problem 1: Finding the inverse Laplace Transforms

Find f(t) for the following Laplace transforms:

F(s) =$\frac{4}{S^2+16}$

Solution:

F(s) =$\frac{4}{s^2+16} = \frac{4}{s^2+4^2}$

f(t)= $\mathcal{L}^{-1}$ **(F(s))**

f(t)= $\mathcal{L}^{-1}\left(\frac{4}{s^2+4^2}\right)$

$\boxed{f(t) = \sin 4t}$

Problem 2:

Find inverse laplace transform for the given function

$$F(s)=\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)$$

Solution:

$$f(t)=\mathcal{L}^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)=\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)]+\mathcal{L}^{-1}\left(\frac{1}{s}\right)+\mathcal{L}^{-1}\left(\frac{s}{s^2-9}\right)$$

$$\boxed{f(t)=e^{3t}+1+\cosh 3t}$$

Problem 3:

Find y(t) for $Y(s)=\left[\frac{1}{s+4}+\frac{1}{s^2+16}+\frac{1}{s^2}\right]$

Solution:

$$y(t)=\mathcal{L}^{-1}\left[\frac{1}{s+4}+\frac{1}{s^2+16}+\frac{1}{s^2}\right]=\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]+\mathcal{L}^{-1}\left[\frac{4}{s^2+16}\right]+\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

$$\boxed{y(t)=e^{-4t}+\sin 4t+t}$$

Problem 4:

Find g(t) for $G(s)=\left[\frac{1}{(s+1)^2+1}\right]$

Solution:

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right)=e^{-t}\,\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)=e^{-t}\sin t$$

$$\boxed{g(t)=e^{-t}\sin t}$$

Problem 5: Finding the inverse Laplace Transforms

Find $\mathcal{L}^{-1}$ for $[F(s)]=\frac{(s+2)}{(s^2+4s+13)}$

Solution:

$$F(s)=\frac{(s+2)}{(s^2+4s+13)}$$

$$\mathcal{L}^{-1}\left(\frac{(s+a)}{(s+a)^2+b^2}\right)=e^{-at}\cos bt$$

$$\mathcal{L}^{-1}\left(\frac{(s+2)}{s^2+4s+4+9}\right)=\mathcal{L}^{-1}\left(\frac{(s+2)}{(s+2)^2+9}\right)=\mathcal{L}^{-1}\left(\frac{(s+2)}{(s+2)^2+3^2}\right)$$

$$\boxed{f(t)=e^{-2t}\cos 3t}$$

Problem 6: Finding the inverse Laplace Transforms

Find $\mathcal{L}^{-1}$ [F(s)] = $\frac{s}{(s+1)^2+1}$

Solution :

$\mathcal{L}^{-1}$ [F(s)] = $\frac{s}{(s+1)^2+1}$ $= \mathcal{L}^{-1}\left(\frac{s+2-2}{(s+1)^2+1}\right)$

$= \mathcal{L}^{-1}\left(\frac{s+2}{(s+1)^2+1}\right) + \mathcal{L}^{-1}\left(\frac{-2}{(s+1)^2+1}\right) = \mathcal{L}^{-1}\left(\frac{s+2}{(s+1)^2+1}\right) - 2\,\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right)$

$= e^{-2t}\,\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - 2\,e^{-2t}\,\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = e^{-2t}.\cos t - 2\,e^{-2t}.\sin t$

$\boxed{f(t) = e^{-2t}\cos t - 2\,e^{-2t}.\sin t}$

Problem 7: Using Partial Fractions

Find f(t) if $\mathcal{L}^{-1}$ [F(s)] = $\frac{(s+4)}{s(s+1)(s^2+5s+6)}$

Solution:

$F(s) = \frac{(s+4)}{s(s+1)(s^2+5s+6)} = \frac{(s+4)}{s(s+1)(s+3)(s+2)}$

$= \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+3)} + \frac{D}{(s+2)} = \frac{A(s+1)(s+3)(s+2)+Bs(s+3)(s+2)+Cs(s+1)(s+2)+Ds(s+1)(s+3)}{s(s+1)(s+3)(s+2)}$

$(s + 4) = A(s + 1)(s + 3)(s + 2) + Bs(s + 3)(s + 2) + Cs(s + 1)(s + 2) + Ds(s + 1)(s + 3)$ ...(1)

Put s=0 in Equation (1)

$A = \frac{2}{3}$

Put s=-1 in Equation (1)

$B = -\frac{3}{2}$

Put s=-3 in Equation (1)

$C = -\frac{1}{6}$

Put s=-2 in Equation (1)

D = 1

$F(s) = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+3)} + \frac{D}{(s+2)} = \frac{\frac{2}{3}}{s} + \frac{\frac{-3}{2}}{(s+1)} + \frac{\frac{-1}{6}}{(s+3)} + \frac{1}{(s+2)}$

Apply $\mathcal{L}^{-1}$ on both sides

$\mathcal{L}^{-1}\,[F(s)] = \mathcal{L}^{-1}\left(\frac{\frac{2}{3}}{s}\right) + \mathcal{L}^{-1}\left(\frac{\frac{-3}{2}}{(s+1)}\right) + \mathcal{L}^{-1}\left(\frac{\frac{-1}{6}}{(s+3)}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+2)}\right)$

$f(t) = \frac{2}{3}(1) - \frac{3}{2}(e^{-t}) - \frac{1}{6}(e^{-3t}) + 1(e^{-2t})$

$$\boxed{f(t) = \frac{2}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t} + e^{-2t}}$$

Problem 8:

Find $\mathcal{L}^{-1}$ [F(s)] = $\frac{(s+2)}{(s-4)(s^2+1)}$

Solution:

$F(s) = \frac{(s+2)}{(s-4)(s^2+1)} = \frac{A}{(s-4)} + \frac{(Bs+C)}{(s^2+1)} = \frac{A(s^2+1)+(Bs+C)(s-4)}{(s-4)(s^2+1)}$

$(s+2) = A(s^2+1) + (Bs+C)(s-4)$ ...... (1)

Put s=4 in Equation (1)

$A = \frac{6}{17}$

Equation (1) becomes

$(s+2) = As^2 + A + Bs^2 - 4Bs + Cs - 4C$ ...... (2)

Equate the co-efficient of 'S^2' term from Equation (2)

0=A+B

A+B=0

$B = -\frac{6}{17}$

Equate the co-efficient of 'S' term from Equation (2)

1=-4B+C

1+4B=C ; $C = -\frac{7}{17}$

$F(s) = \frac{A}{(s-4)} + \frac{(Bs+C)}{(s^2+1)} = \frac{\frac{6}{17}}{(s-4)} + \frac{(-\frac{6}{17}s - \frac{7}{17})}{(s^2+1)}$

Apply $\mathcal{L}^{-1}$ on both sides

$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left(\frac{\frac{6}{17}}{(s-4)}\right) + \mathcal{L}^{-1}\left(\frac{(-\frac{6}{17}s - \frac{7}{17})}{(s^2+1)}\right)$

$$\boxed{f(t) = \frac{6}{17}(e^{4t}) - \frac{6}{17}(\cos t) - \frac{7}{17}(\sin t)}$$

Problem 9:

Find Laplace Inverse for the following function:

$Y = \frac{1}{(s-2)(s+1)(s+3)}$

Solution:

Given:

$Y = \frac{1}{(s-2)(s+1)(s+3)}$

Apply Laplace Inverse

$Y = \mathcal{L}^{-1} \frac{1}{(s-2)(s+1)(s+3)}$

By using Partial Fraction

$$\frac{1}{(s-2)(s+1)(s+3)} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s+3} = \frac{A\,(s+1)(s+3) + B(s-2)(s+3) + C(s-2)(s+1)}{(s-2)(s+1)(s+3)}$$

$$\frac{1}{(s-2)(s+1)(s+3)} = \frac{A\,(s+1)(s+3) + B(s-2)(s+3) + C(s-2)(s+1)}{(s-2)(s+1)(s+3)}$$

1= A(s+1) (s+3) + B(s-2) (s+3) + C(s-2) (s+1) ..... (1)

Put s = 2 in equation (1)

1= A (2+1) (2+3) + B (0) + C(0)

1= A (3) (5)

1=15A

$A = \frac{1}{15}$

Put S = -1 in equation (1)

1= A (-1+1) (-1+3) + B (-1-2) (-1+3)+ C(0)

1= A (0) + B (-3) (2)

1= -6B

$B = -\frac{1}{6}$

Put S = -3 in equation (1)

1= A (0) + B(0)+ C(-3-2)(-3+1)

1= C (-5) (-2)

1= 10C

$C = \frac{1}{10}$

$Y(s) = = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s+3}$

$Y(s) = = \frac{1/15}{s-2} - \frac{1/6}{s+1} + \frac{1/10}{s+3}$

Apply $\mathcal{L}^{-1}$ on both sides

$$\mathcal{L}^{-1}\,(Y(s)) = \frac{1}{15}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{10}\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$$

$y(t) = \frac{1}{15}(e^{2t}) - \frac{1}{6}(e^{-t}) + \frac{1}{10}(e^{-3t})$

$$\boxed{y(t) = \frac{e^{2t}}{15} - \frac{e^{-t}}{6} + \frac{e^{-3t}}{10}}$$

Problem 10:

$Y(s) = \mathcal{L}^{-1}\ \frac{1}{s(s+12)}$

$\frac{1}{s(s+12)} = \frac{A}{s} + \frac{B}{s+12} = \frac{A\,(s+12) + Bs}{s(s+12)}$

$\frac{1}{s(s+12)} = \frac{A\,(s+12) + Bs}{s(s+12)}$

$1 = A\,(s + 12) + Bs$ ...... (1)

Put s = 0

$1 = A\,(0 + 12) + B\,(0)$

$1 = 12A$

$A = \frac{1}{12}$

Put S = -12

$1 = A\,(-12 + 12) + B\,(-12)$

$1 = A\,(0) + B\,(-12)$

$1 = -12B$

$B = -\frac{1}{12}$

$\frac{A}{s} + \frac{B}{s+12} = \frac{\left(\frac{1}{12}\right)}{s} + \frac{\left(-\frac{1}{12}\right)}{(s+12)} = \frac{1}{12}\left(\frac{1}{s}\right) - \frac{1}{12}\left(\frac{1}{s+12}\right)$

Apply $\mathcal{L}^{-1}$ on both sides

$= \frac{1}{12}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{12}\mathcal{L}^{-1}\left(\frac{1}{s+12}\right)$

$= \frac{1}{12}(1) - \frac{1}{12}(e^{-12t})$

$$\boxed{y(t) = \frac{1}{12} - \frac{e^{-12t}}{12}}$$

Problem 11: Find the inverse Laplace Transforms

Find $\mathcal{L}^{-1}$ $[F(s)] = \frac{(s+2)}{(s+2s+5)}$

Solution:

$F(s) = \frac{(s+2)}{(s+2s+5)}$

$\mathcal{L}^{-1} \frac{(s+a)}{(s+a)^2+b^2} = [e^{-at}\cos bt]$

$\mathcal{L}^{-1}\left(\frac{(s+1+1)}{s^2+2s+1+4}\right) = \mathcal{L}^{-1}\left[\frac{(s+1)}{(s+1)^2+4} + \frac{(1)}{(s+1)^2+4}\right]$

$= \mathcal{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+2^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$

$\boxed{f(t) = e^{-t}\cos 2t + e^{-t}\sin 2t}$

1.6 TRANSFER FUNCTION:

The transfer function of a system is defined as the ratio of Laplace transform of Output to the Laplace transform of input with zero initial conditions.

$$\boxed{\textbf{Transfer function} = \frac{\text{Laplace Transform of Output}}{\text{Laplace Transform of Output}} \Big|_{\text{with zero initial condition}}}$$

The transfer function can be obtained by taking Laplace transform of the differential equation governing the system with zero initial conditions and rearranging the resulting algebraic equations to get the ratio of output to input.

1.7 ORDER AND TYPE OF A TRANSFER FUNCTION:

1.7.1 ORDER OF A SYSTEM:

The input and output relationship of a control system can be expressed by n^{th} order differential equation is given below.

$$a_0 \frac{d^n}{dt^n}p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}}p(t) + a_2 \frac{d^{n-2}}{dt^{n-2}}p(t) + \ldots\ldots + a_{n-1}\frac{d}{dt}p(t) + a_n\, p(t) = b_0 \frac{d^m}{dt^m}q(t)$$
$$+ b_1 \frac{d^{m-1}}{dt^{m-1}}q(t) + b_2 \frac{d^{m-2}}{dt^{m-2}}q(t) + \ldots\ldots + b_{m-1}\frac{d}{dt}q(t) + b_m\, q(t)$$

where, $p(t)$ = Output / Response ; $q(t)$ = Input / Excitation.

The order of the system is given by the order of the differential equation governing the system. If the system is governed by n^{th}order differential equation, then the system is called order of a system. Alternatively, the order can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in s, as shown in equation.

$$\text{Transfer function, } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots\dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n}$$

where, P(s) = Numerator polynomial

Q(s) = Denominator polynomial

The order of the system is given by the maximum power of s in the denominator polynomial, Q(s).

$$\text{Here, } Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots\dots\dots + a_{n-1} s + a_n.$$

Now, n is the order of the system

When n == 0, the system is zero order system.

When n = 1, the system is first order system.

When n = 2, the system is second order system and so on.

[*Note: The order can be specified for both open loop system and closed loop system.]*

1.7.2 TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(s) H(s). The number of poles of the transfer function lying at the origin decides the type number of the system. In general, if N is the number of poles at the origin then the type number is N.

The loop transfer function can be expressed as a ratio of two polynomials in S.

$$G(s)\, H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\dots\dots\dots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\dots\dots}$$

where, Z1, Z2, Z3…. are zeros of transfer function

P1, P2, P3 ….. are poles of transfer function

K = Constant

N == Number of poles at the origin

The value of N in the denominator polynomial of loop transfer function shown in equation decides the type number of the system.

If N = 0, then the system is type - 0 system

If N = 1, then the system is type - 1 system, and so on.

1.8 POLE – ZERO PLOT

1.8.1 POLES

Poles are those values of s which make F(s) tend to infinity.

For example: $F(s) = \frac{(s+z_1)(s+z_2)(as^2+bs+c)}{s^2(s+p_1)(s+p_2)(As^2+Bs+C)}$

Then we have a double pole origin. Two simple poles at s = $-p_1$, s = $-p_2$ and one pair of poles at $s = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$

This may be complex, real or imaginary depending upon the numerical values of A, B and C.

At s = 0, F(s) = ∞, a double pole at s = 0

At s = $-p_1$, F(s) = ∞, a simple pole at s = $-p_1$

At s = $-p_2$, F(s) = ∞, a simple pole at s = $-p_2$

At s = $\frac{-B\pm\sqrt{B^2-4AC}}{2A}$, F(s) = ∞ the roots may be real, imaginary, complex conjugate.

1.8.2 ZEROS

The zeros of F(s) are those values of s which make F(s) tend to zero.

For the function, F(S), the zeros are at s = $-z_1$, s = $-z_2$ and s= $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

At s = $-z_1$, F(s) = 0, a simple zero at s = $-z_1$

At s = $-z_2$, F(s) = 0, a simple zero at s = $-z_2$

At s= $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, F(s) = 0 the roots may be real, imaginary, complex conjugate.

EXAMPLE 1:

1. Draw the pole and zero configuration for the following functions

(i) $\mathbf{F(s) = \frac{4(s+4)}{s(s+3)(s^2+2s+2)}}$

Solution:

a) To Find Poles:

$$\mathbf{s(s+3)(s^2+2s+2) = 0}$$

At s = 0, F(s) = ∞, a simple pole at origin or s = 0

At s = -3, F(s) = ∞, a simple pole at s = -3

$s^2 + 2s + 2 = 0$; By using quadratic equation: $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, we get

$$\frac{-2\pm\sqrt{2^2-4(1)(2)}}{2(1)} = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2} = \frac{-2\pm\sqrt{j^2 2^2}}{2} = \frac{-2\pm j2}{2} = \frac{2(-1\pm j)}{2} = -1 \pm j$$

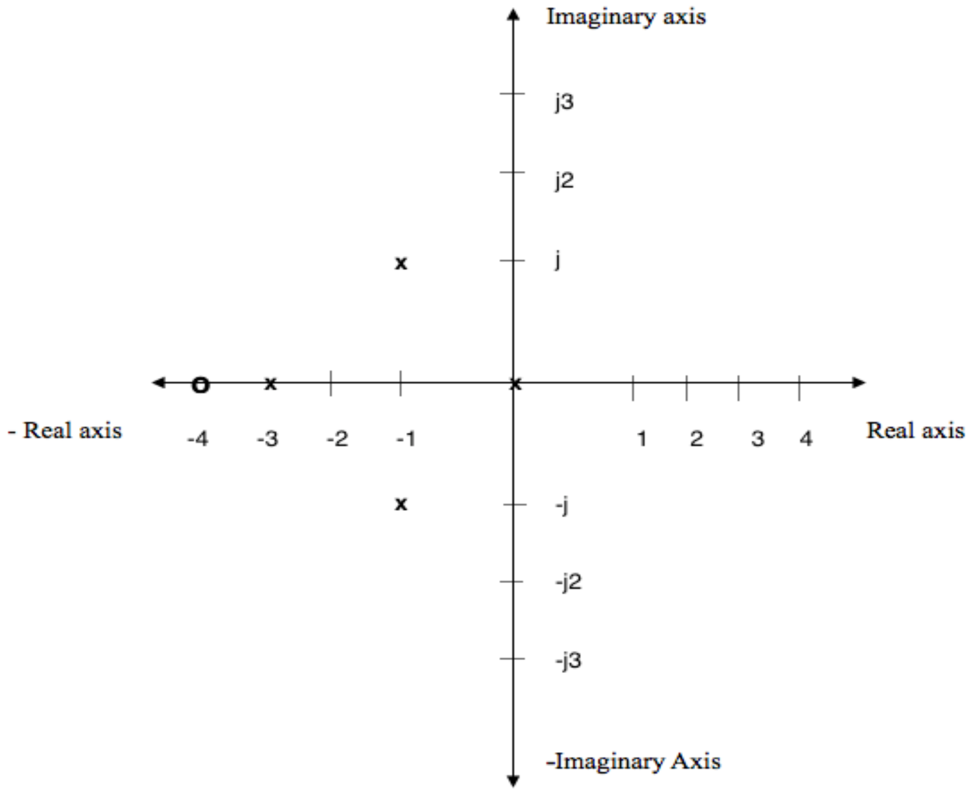
At s = -1+j, F(s) = ∞, a complex conjugate pole at s = -1+j

At s = -1-j, F(s) = ∞, a complex conjugate pole at s = -1-j

b) To Find Zeros:

4(s+4) = 0

At s = -4, F(s) = 0, a simple zero at s = -4



<u>RESULT</u>

The poles are {0, -3, -1+j, -1-j}

The zeros are {-4}

(ii) $\mathbf{F(s) = \frac{(s+1)(s+5)}{s(s+2)(s+4)}}$

Solution:

a) To Find Poles:

$$\mathbf{s(s+2)(s+4) = 0}$$

At s = 0, F(s) = ∞, a simple pole at origin or s = 0

At s = -2, F(s) = ∞, a simple pole at s = -2

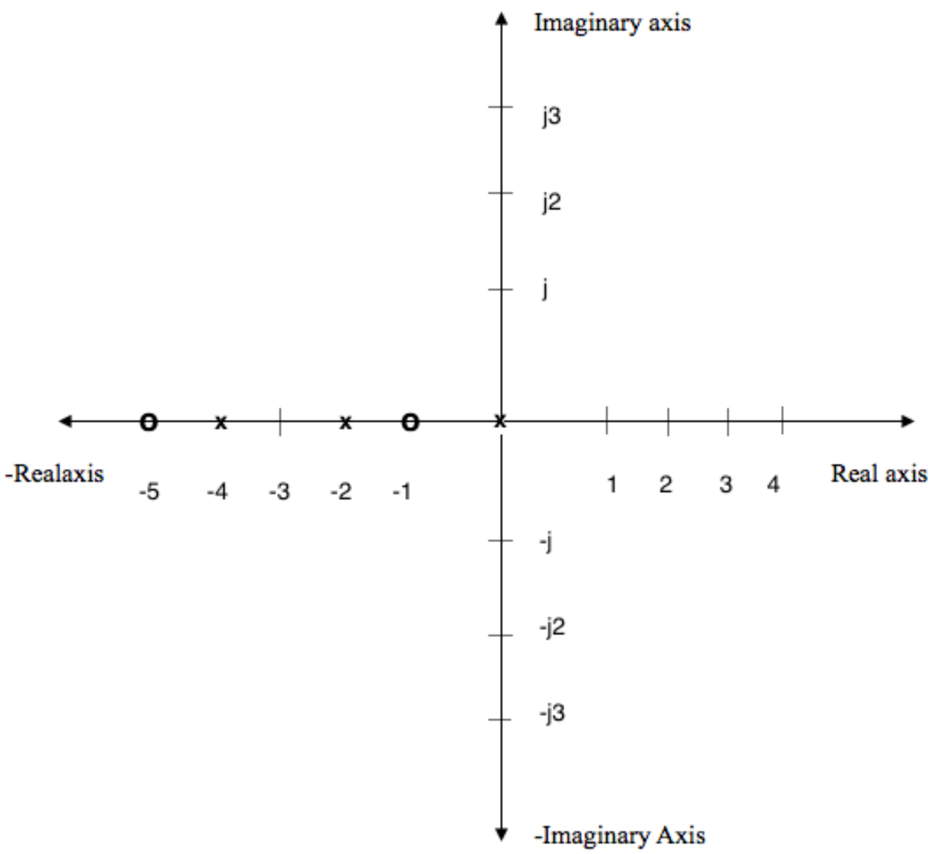
At s= -4, F(s) = ∞, a simple pole at s = -4

b) To Find Zeros:

$$\mathbf{(s+1)(s+5) = 0}$$

At s = -1, F(s) = 0, a simple zero at s = -1

At s = -5, F(s) = 0, a simple zero at s = -5



RESULT

The poles are {0, -2, -4} **The zeros are {-1, -5}**

(iii) $F(s) = \frac{s^3(s+1)}{(s+2)^2(s^2+4)}$

Solution:

a) To Find Poles:

$$(s+2)^2(s^2+4)=0$$

$(s+2)^2 = 0\,; (s+2) = 0(\text{twice}); s = -2$

At s = -2, F(s) = ∞, a double pole at s = -2

$s^2+4=0;\ s^2 = -4;\ s^2 = j^2 2^2 = \pm j2$

At s = j2, F(s) = ∞, an imaginary pole at s = j2

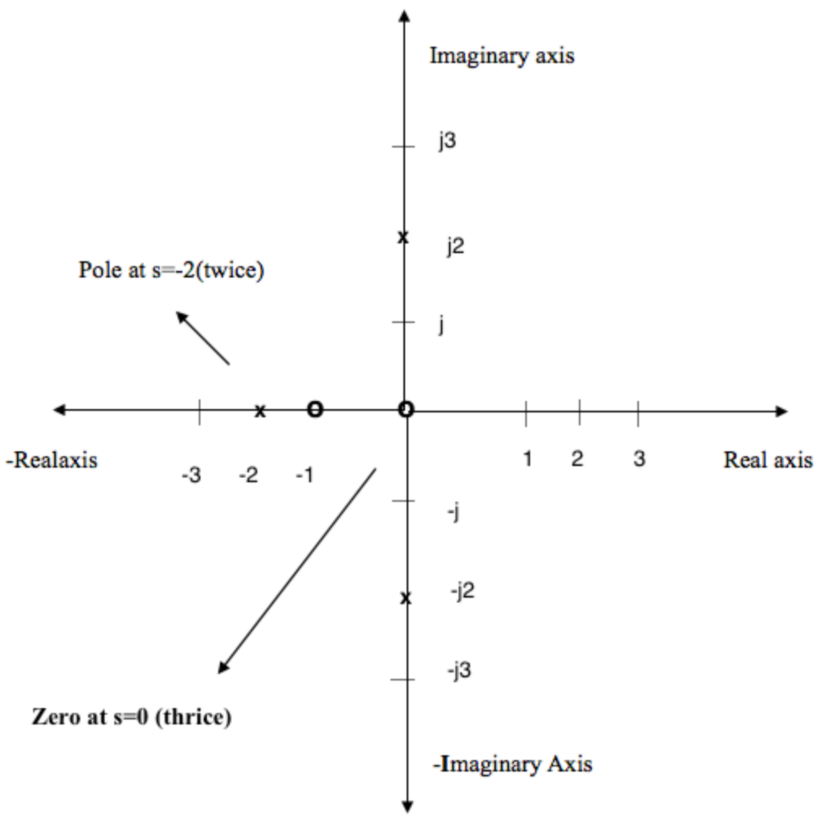
At s= -j2, F(s) = ∞, an imaginary pole at s = -j2

b) To Find Zeros:

$$s^3(s+1)=0$$

At s = 0, F(s) = 0, a triple zero at s = 0

At s = -1, F(s) = 0, a simple zero at s = -1



RESULT

The poles are {-2, -j2, j2}
The zeros are {0, -1}

(iv) $F(s) = \frac{(s^2+9)(s^2+4s+13)}{s^2(s^3+10s^2+29s+20)}$

Solution:

a) To Find Poles

$$s^2(s^3+10s^2+29s+20)=0$$

On factorisation of $(s^3+10s^2+29s+20)$, we get: $(s+1)(s^2+9s+20) = (s+1)(s+5)(s+4)$

At s = 0, F(s) = ∞, a double pole at origin or s = 0

At s = -1, F(s) = ∞, a simple pole at s = -1

At s = -4, F(s) = ∞, a simple pole at s = -4

At s = -5, F(s) = ∞, a simple pole at s = -5

b) To Find Zeros

$$(s^2+9)(s^2+4s+13)=0$$

$(s^2+9)=0;\ s^2 = -9;\ (s^2 = (j^2)\ (3^2) = \pm j3$

At s = -j3, F(s) = 0, an imaginary zero at s = -j3

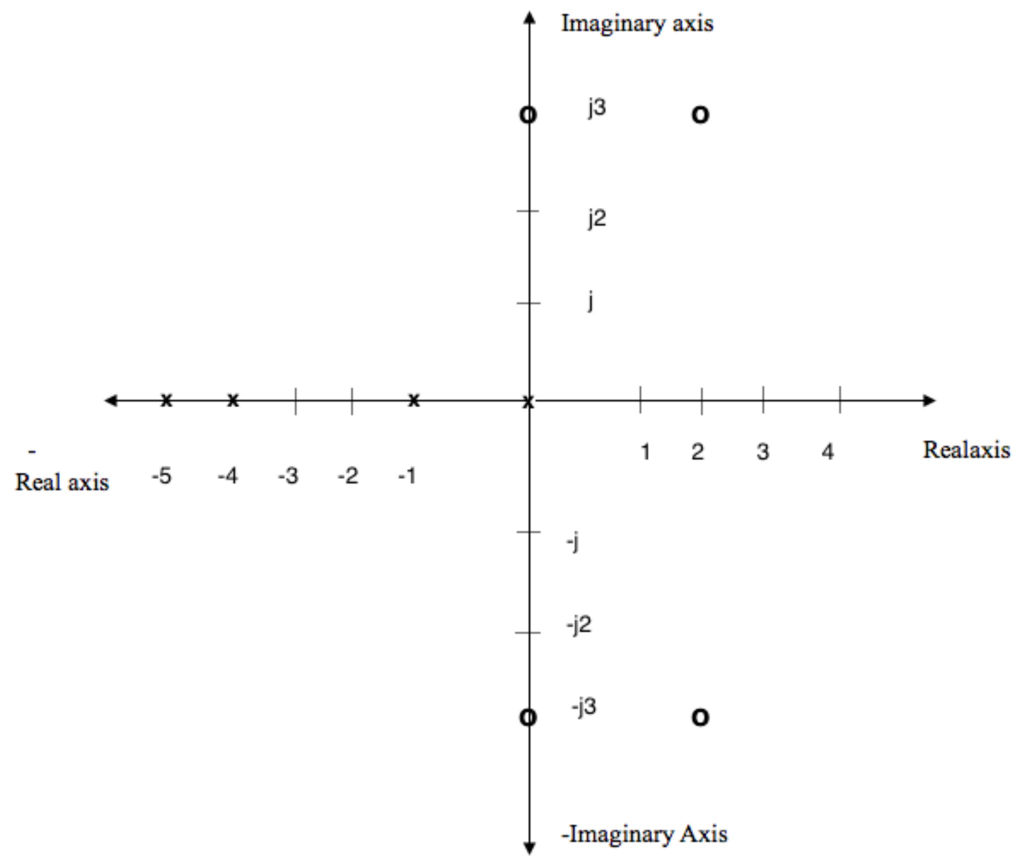
At s = -j3, F(s) = 0, an imaginary zero at s = -j3

$(s^2+4s+13)=0$;By using quadratic equation$\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$, we get

$$\frac{-4\pm\sqrt{4^2-4(1)(13)}}{2(1)} = \frac{-4\pm\sqrt{16-52}}{2} = \frac{-4\pm\sqrt{-36}}{2} = \frac{-4\pm\sqrt{j^2 6^2}}{2} = \frac{-4\pm j6}{2} = \frac{2(-2\pm j3)}{2} = -2 \pm j3$$

At s = -2+j3, F(s) = 0, an imaginary zero at s = -2+j3

At s = -2-j3, F(s) = 0, an imaginary zero at s = -2-j3



RESULT

The poles are { 0, -1, -4, -5 }
The zeros are { -j3, j3, -2-j3, -2+j3 }

1.9 TRANSFER FUNCTION OF A MECHANICAL TRANSLATIONAL SYSTEM

Translation is defined as a motion that takes place along a straight line. The three basic elements involved in translational motion are ***mass, spring and dash-pot***. These three elements represents three essential phenomena which occur in various ways in mechanical systems.

The weight of the mechanical system is represented by the element ***mass*** and it is assumed to be concentrated at the centre of the body. The elastic deformation of the body can be represented by a ***Spring***. The friction existing in rotating mechanical system can be represented by the ***dash-pot***.

When a Force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by ***Newton's Second law of motion***. For translational systems, it states that the sum of forces acting on a body is zero.

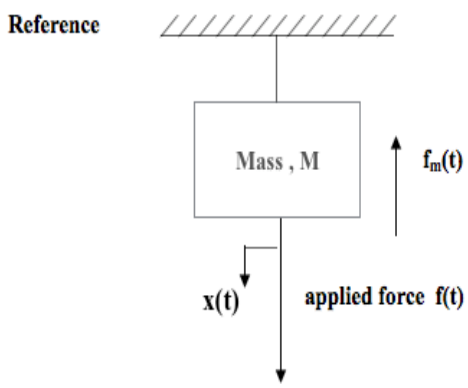
1.9.1 List Of Symbols Used In Mechanical Translational System

x = Displacement, m.
v = dx/dt = Velocity, m/sec.
a = $dv/dt = d^2x/dt^2$ = Acceleration, m/sec^2.
f = Applied Force, N(Newtons).
f_m = Opposing force offered by mass of the body, N.
f_k = Opposing force offered by the elasticity of the body(spring), N.
f_b = Opposing force offered by the friction of the body(dash-pot), N.
M = Mass, kg.
K= Stiffness of spring, N/m.
B = Viscous force co-efficient, N-sec/m.

1.9.2 Mass(M):

When a force is applied, the mass will offer an opposing force which is proportional to the acceleration of the body.

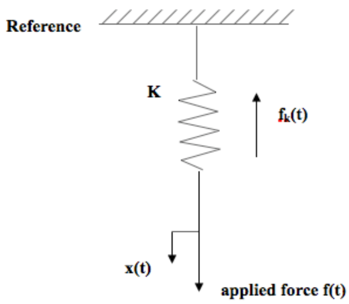
$f_m(t)$ = Ma; $\quad f_m(t)=M\frac{d^2x(t)}{dt^2}$ $\quad$ where, $a=\frac{d^2x(t)}{dt^2}$



1.9.3 Stiffness (K):

When a force is applied, the spring will offer an opposing force which is proportional to displacement of the body.

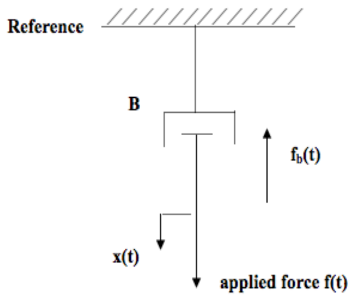
$$f_k(t)=K.x(t)$$



1.9.4 Viscous Friction (B):

When a force is applied, the dash-pot will offer an opposing force which is proportional to velocity of the body.

$$f_b(t)=B.\frac{dx(t)}{dt}$$



1.9.5 GUIDELINES TO DETERMINE THE TRANSFER FUNCTION OF MECHANICAL TRANSLATIONAL SYSTEM

- In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system. The nodes are meeting point of elements. Generally, the nodes are mass elements in the system. In some cases, the nodes may be without mass element.
- The linear displacement of the masses (nodes) are assumed as x_1, x_2, x_3, etc., and assign a displacement to each mass(node). The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration.
- Draw the free body diagrams of the system. The free body diagram is obtained by drawing each mass separately and then marking all the forces acting on that mass(node). Always the opposing force acts in a direction of the applied force. Hence the displacement, velocity and acceleration of the mass will be in the direction of the applied force. If there is no applied force then the displacement, velocity and acceleration of the mass will be in the direction opposite to that of opposing force.
- For each free body diagram, write one differential equation by equating the sum of applied forces to the sum of opposing forces.
- Take Laplace transform of differential equation to convert them to algebraic equation. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the ratio between output variable and input variable. This ratio is the transfer function of the system.

Note: Laplace transform of x(t) = $\mathcal{L}$ {x(t)} = X(s) Laplace transform of = $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\}$ = s X(s) Laplace transform of $\frac{d^2x(t)}{dt^2} = \mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = s^2X(s)$

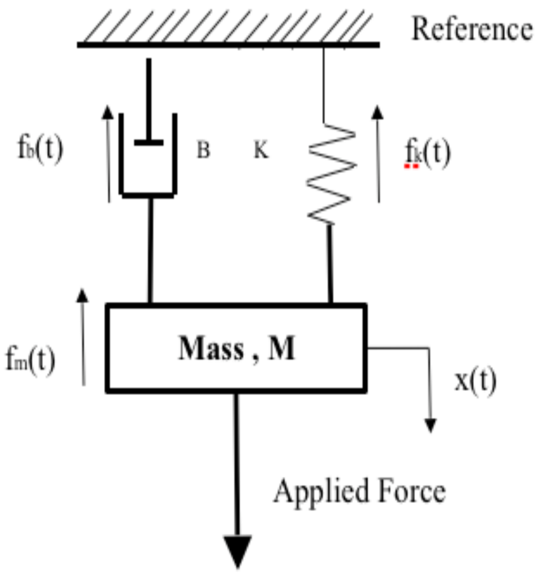


Fig 1.5: Mechanical Translational System

The applied force f(t) is resisted by forces $f_m(t)$, $f_k(t)$, $f_b(t)$, so the equation of motion is

$$\mathbf{f(t) = f_m(t) + f_b(t) + f_k(t)}$$

$$\mathbf{M\,\frac{d^2x(t)}{dt^2} + B.\frac{dx(t)}{dt} + K.x(t)}$$

Taking Laplace transform on both sides with zero initial conditions

$F(s) = Ms^2X(s) + BsX(s) + KX(s)$

$F(s) = X(s)[Ms^2 + Bs + K]$

Transfer Function = $\frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K}$

1.10 TRANSFER FUNCTION OF ELECTRICAL SYSTEMS USING R L C

The models of electrical systems can be obtained by using resistor, capacitor and inductors. For modelling electrical systems, the electrical network or equivalent circuit is formed by using R, L and C and voltage or current source.

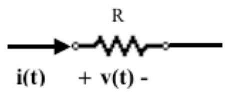
The differential equations governing the electrical systems can be formed by writing

Kirchhoff's current law equations by choosing various nodes in the network or Kirchhoff's voltage law equations by choosing various closed path in the network.

The transfer function can be obtained by taking Laplace transform of the differential equations and rearranging them as a ratio of output to input.

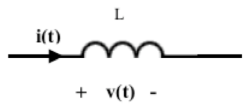
Current -Voltage relation of Resistor:

Voltage across the resistor, $v(t) = R\, i(t)$



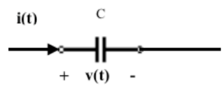
Current -Voltage relation of Inductor:

Voltage across the Inductor, $v(t) = L \frac{d}{dt} i(t)$



Current -Voltage relation of Capacitor:

Voltage across the Capacitor, $v(t) = \frac{1}{C}\int i(t)dt$



1.11 TRANSFER FUNCTION OF AN RC NETWORK:

Voltage across the resistor, $v(t) = R\, i(t)$

Voltage across the Capacitor, $v(t) = \frac{1}{C}\int i(t)dt$

According to Kirchhoff's second law,

$$R\, i(t) + \frac{1}{C}\int i(t)dt = e_1 \qquad \text{...... (1)}$$

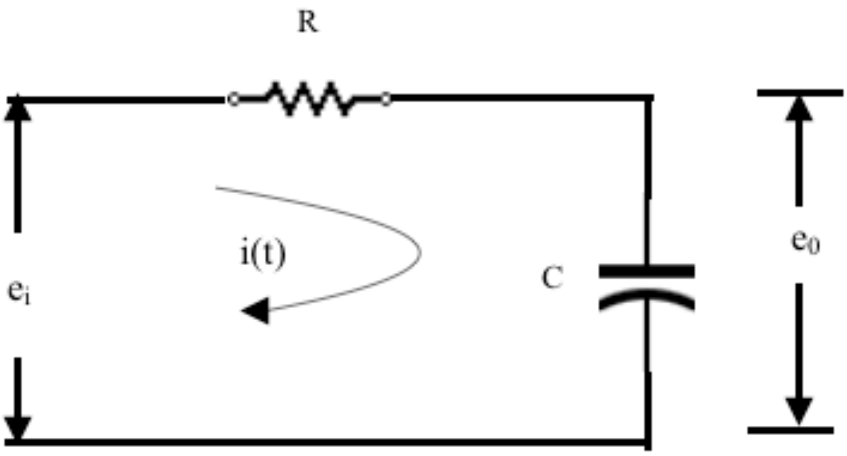


Fig 1.6: R C Network

On Taking Laplace Transform on both sides, we get

$R\, I(s) + \frac{1}{C}\frac{I(s)}{s} = E_i(s)$

$I(s) \left(R + \frac{1}{Cs}\right) = E_i(s)$

$I(s) = \frac{E_i(s)}{\left(R+\frac{1}{Cs}\right)} = \frac{E_i(s)}{\frac{RCs+1}{Cs}} = \frac{CsE_i(s)}{RCs+1}$ ...... (2)

We have, $e_0 = \frac{1}{C}\int i(t)dt$

On taking Laplace transform on both sides,

$E_0(s) = \frac{1}{C}\frac{I(s)}{s}$ ...... (3)

Substitute I(s) from equation 2 in equation 3,

$E_0(s) = \frac{\frac{CsE_i(s)}{RCs+1}}{Cs} = \frac{CsE_i(s)}{RCs+1} \times \frac{1}{Cs} = \frac{E_i(s)}{RCs+1}$

$\frac{E_0(s)}{E_i(s)} = \frac{1}{RCs+1} = \frac{1}{\tau s+1}$

$RC = \tau$ (time Constant)

Thus, the Transfer Function of the RL Circuit is $\mathbf{\frac{E_0(s)}{E_i(s)} = \frac{1}{\tau s+1}}$

1.12 TRANSFER FUNCTION OF RLC CIRCUIT:

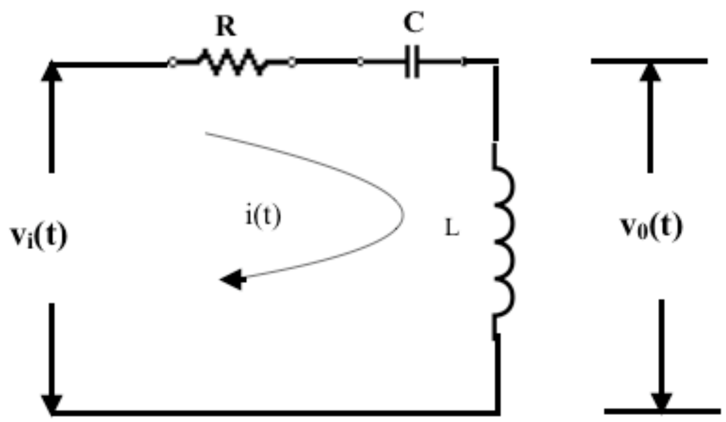


Fig 1.7: R L C Network

Voltage across the resistor, $v(t) = R\, i(t)$

Voltage across the Capacitor, $v(t) = \frac{1}{C}\int i(t)dt$

Voltage across the Inductor, $v(t) = L\frac{d}{dt}i(t)$

According to Kirchhoff's Voltage law, $V_R + V_C + V_L = V_i(t)$

$$R\, i(t) += \frac{1}{C}\int i(t)dt + L\frac{d}{dt} i(t) = V_i(t) \qquad \ldots\ldots (1)$$

On taking laplace transform,

$$R\, I(s) + \frac{1}{C}\frac{I(s)}{s} + Ls\, I(s) = V_i(s)$$

$$I(s)\, [\, R + Ls + \frac{1}{Cs}\,] = V_i(s)$$

$$I(s) = \frac{V_i(s)}{R + Ls + \frac{1}{Cs}} = \frac{V_i(s)}{\frac{RCs + LCs^2 + 1}{Cs}} = \frac{Cs\, V_i(s)}{RCs + LCs^2 + 1} \qquad \ldots\ldots (2)$$

$$V_0(t) = L\frac{d}{dt} i(t)$$

On taking Laplace transform on both sides,

$$V_0(s) = Ls\, I(s) \qquad \ldots\ldots (3)$$

Substitute I(s) from equation 2 in equation 3,

$$V_0(s) = Ls\frac{Cs\, V_i(s)}{RCs + LCs^2 + 1}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{LCs^2}{RCs + LCs^2 + 1}$$

Thus, the Transfer Function of the R L C Circuit is $\frac{V_0(s)}{V_i(s)} = \frac{LCs^2}{RCs + LCs^2 + 1}$

PART – A

1.Define Control system.

2.List the Advantages of Open loop system

3.What are the Disadvantages of open loop system?

4.What is SISO & MIMO?

5.What is Linear system?

6.Define Laplace Transform.

7.Define Laplace Inverse.

8.What is Transfer function?

9.Define Poles.

10.Define Zeros.

11.Find the Laplace Transform of sinωt.

PART – B

1.Classify Control system.

2.Define Open loop system and draw its Block diagram.

3.Define Closed loop system with neat diagram.

4.Distinguish between open loop & closed loop system.

5. What is Linear & Non-linear system?

6.What is continuous & discontinuous system?

7.What is Order of a system?

8.Define Type-Number of control system.

9.Find Laplace transform of $e^{6t}\cosh 2t$

10.Find $\mathcal{L}^{-1}\left(\frac{(s-1)}{(s-1)^2-5^2}\right)$.

PART – C:

1.f(t) = (e^{2t}+3 e^{-5t}) Ans: $\frac{1}{s-2}+3.\frac{1}{s+5}$

2.g(t) =(2+5cos t) Ans: $\frac{2}{s}+5.\frac{s}{s^2+1}$

3.y(t)=(sinh 6t+3t) Ans: $\frac{6}{s^2-36}+\frac{3}{s^2}$

4.y(t) =5e^{-8t}+cosh3t Ans: $\frac{5}{s+8}+\frac{S}{s^2-9}$

5.h(t) = $7t^3$+5sin3t Ans: $7.\frac{6}{s^4}+5.\frac{3}{s^2+9}$

6. $\frac{1}{(s+1)(s^2+2s+2)}$ Ans: e^{-t} (1-cos t)

7. $\frac{1}{(s+1)(s+3)}$ Ans: $\frac{1}{4}$ (e^{t} - e^{-3t})

8. $\frac{1}{(s+1)(s^2+1)}$ Ans : $\frac{1}{2}$ ($\sin t - \cos t + e^{-t}$)

9. $\frac{20}{(s-4)(s+6)}$ Ans : ($2e^{4t}$ -2 e^{-6t})

10. $\frac{1}{s(s+1)^2}$ Ans: (1-te^{t} - e^{-t})

11. $\frac{s+2}{s(s+3)(s+4)}$ Ans: ($\frac{1}{6}+\frac{1}{3}e^{-3t}-\frac{1}{2}e^{-4t}$)

UNIT – II

BLOCK DIAGRAM AND SIGNAL FLOW GRAPH PRESENTATION

2.1 BLOCK DIAGRAM:

A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components. The elements of a block diagram are

- Block
- Branch point
- Summing point

2.1.1 BLOCK:

The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signal. Figure shows the block diagram of the functional block.

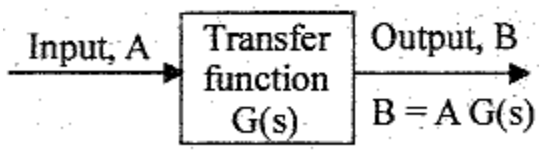


Fig.2.1 Block

The arrowhead pointing towards the block indicates the input, and the arrow head leading away from the block represents the output. Such arrows are referred to as signals. The. Output signal from the block is given by the product of input signal and transfer function in the block.

2.1.2 SUMMING POINT:

Summing points are used to add two or more signals in the system. A circle with a cross is the symbol that indicates a summing operation. The plus or minus sign at each arrowhead indicates whether the signal to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

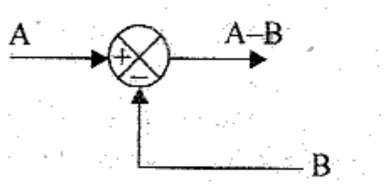


Fig.2.2 Summing Point

2.1.3 BRANCH POINT:

A Branch point is a point from which the signal from a block concurrently to other blocks or summing points.

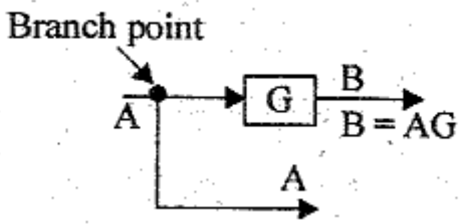


Fig.2.3 Branch Point

2.2 BLOCK DIAGRAM REDUCTION:

The block diagram can be reduced to find the overall transfer function of the system. The following rules can be used for block diagram reduction. The rules are framed such that any modification made on the diagram does not alter the input output relation.

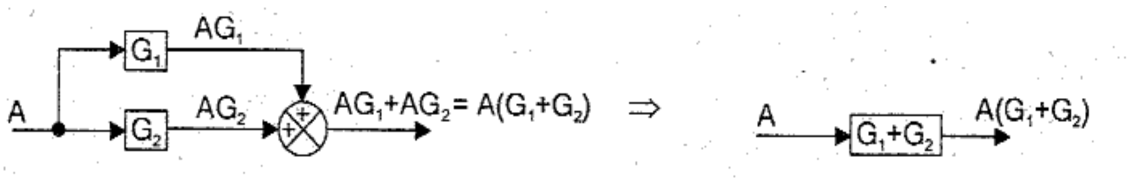
2.3 ADVANTAGES OF BLOCK DIAGRAM:

- The Functional operation of the system can be observed from Block diagram.
- Block diagram gives the Information about performance of the system.
- Block diagram is used for Analysis and Design of control system.
- It is very simple to construct the Block diagram for complicated system.

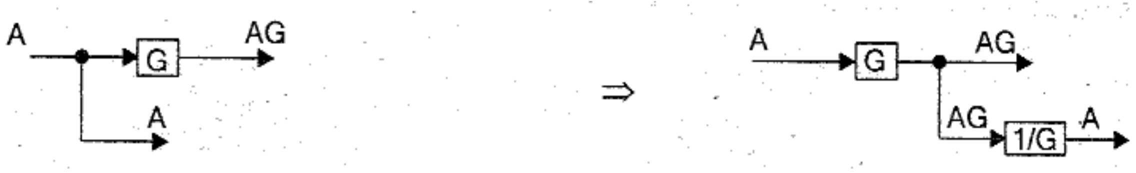
2.4 RULES OF BLOCK DIAGRAM

RULE 1. Combining the Blocks in Cascade

RULE 2. Combining Parallel Blocks (or) Combining Feed Forward Path

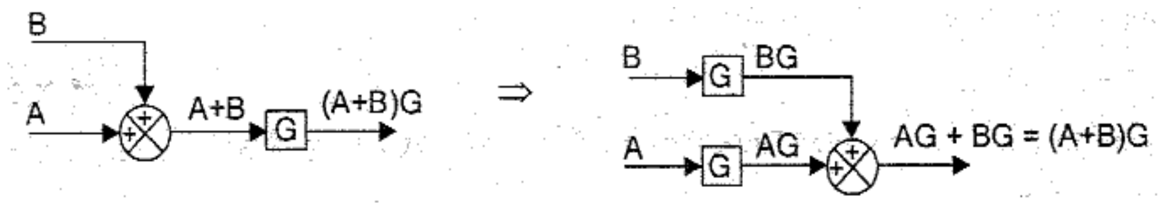


RULE 3: Moving the Branch Point Ahead of the Block

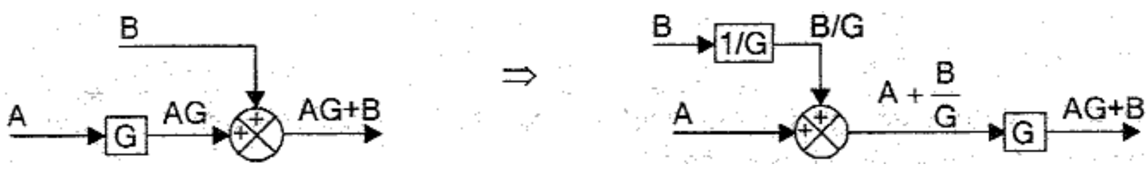


RULE 4: Moving the Branch Point Before the Block

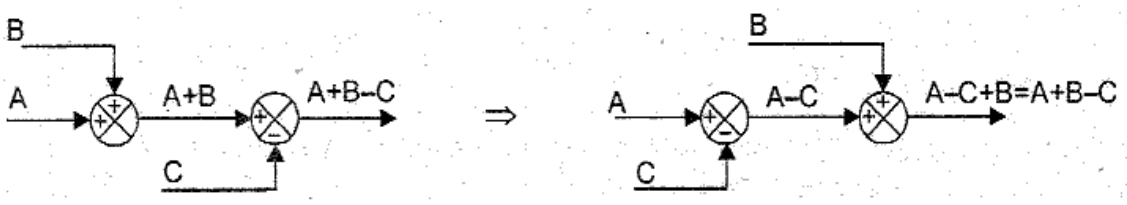
RULE 5: Moving the Summing Ahead of the Block



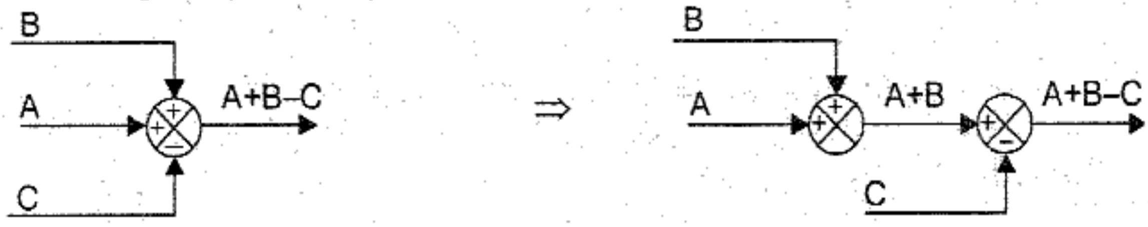
RULE 6: Moving the Summing Before of the Block



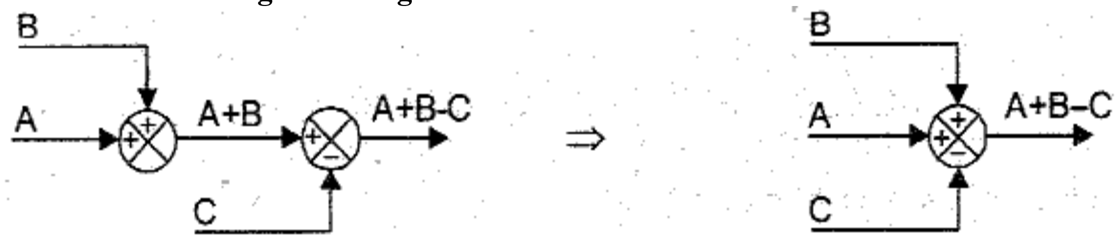
RULE 7: Interchanging Summing Point



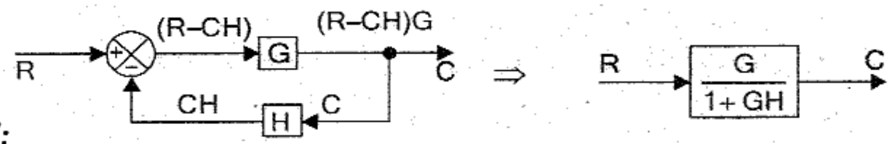
RULE 8: Splitting Summing Point



RULE 9: Combining Summing Point



RULE 10: Elimination of Negative Feedback Loop

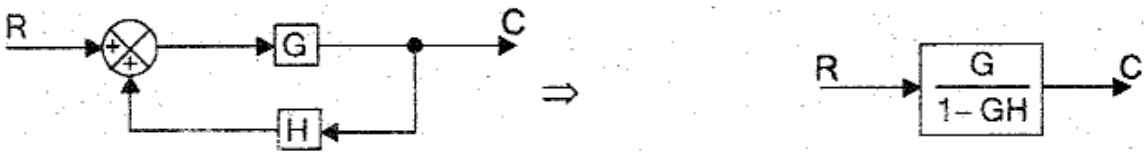


Proof:

$C = (R - CH)G \Rightarrow C = RG - CHG \Rightarrow C + CHG = RG$

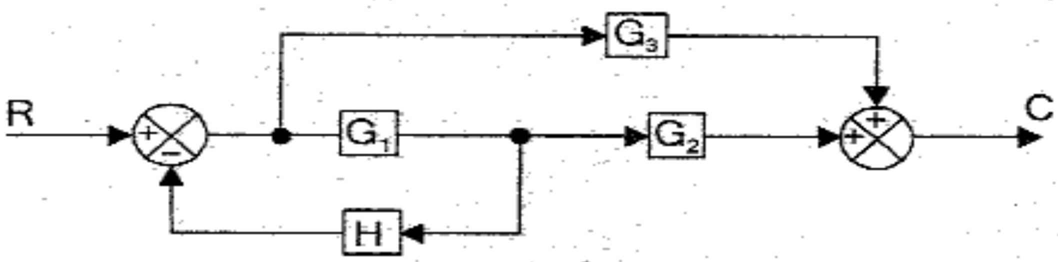
$\therefore C(1 + HG) = RG \Rightarrow \frac{C}{R} = \frac{G}{1+GH}$

RULE 11: Elimination of Positive Feedback Loop



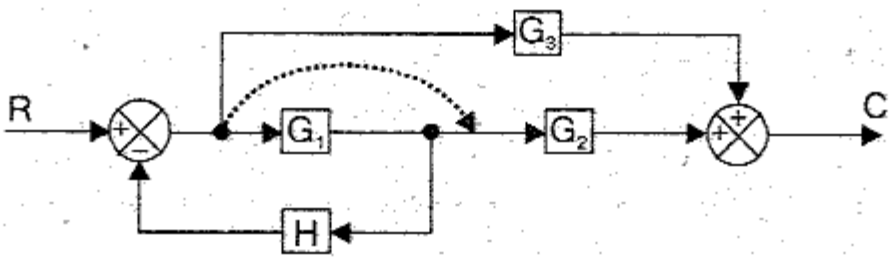
2.5 EXAMPLES OF BLOCK DIAGRAM REDUCTION TECHNIQUE:
EXAMPLE 1:

Reduce the Block diagram shown in figure and find C/R.

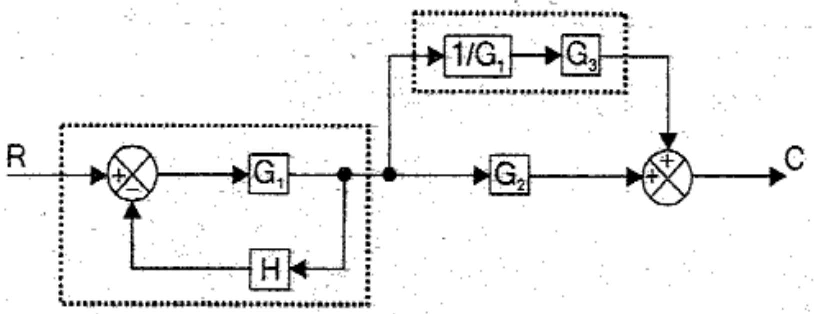


SOLUTION:

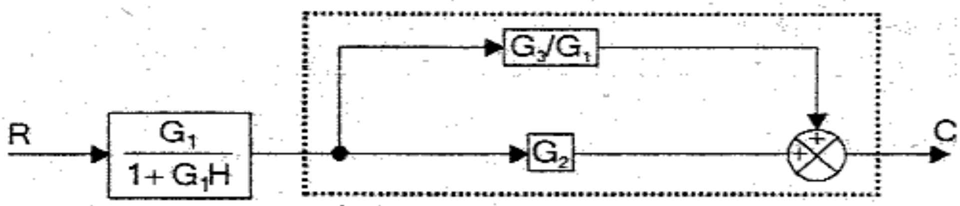
Step1: Move the Branch point after the Block



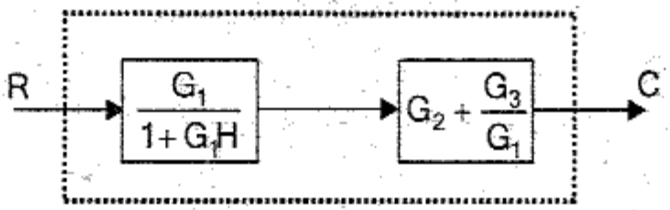
Step2: Eliminate the feedback path and combining blocks in cascade



Step3: Combining parallel blocks



Stsp4: Combining blocks in Cascade



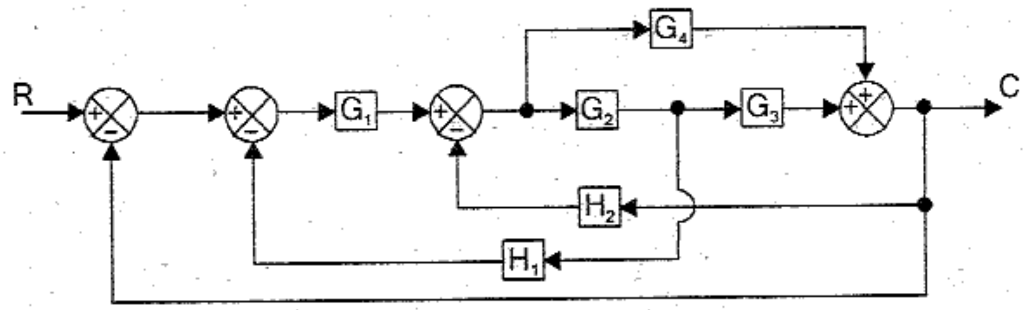
$$\frac{C}{R} = \left(\frac{G_1}{1+G_1H}\right)\left(G_2 + \frac{G_3}{G_1}\right) = \left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2+G_3}{G_1}\right) = \frac{G_1G_2+G_3}{1+G_1H}$$

RESULT:

The Overall Transfer function of the system, $\frac{C}{R} = \frac{G_1G_2+G_3}{1+G_1H}$

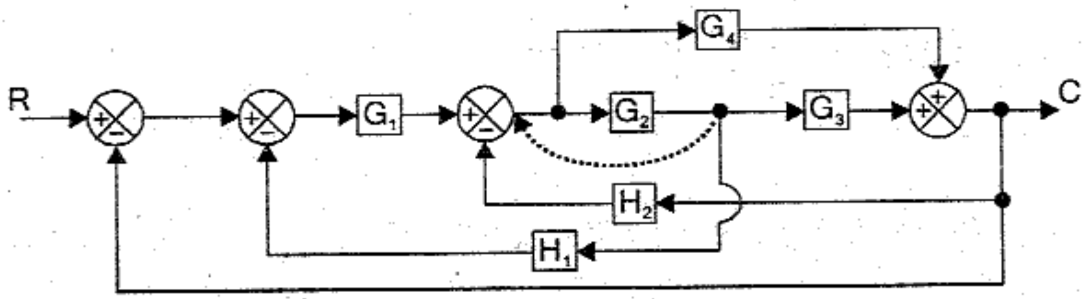
EXAMPLE 2:

Using Block diagram reduction technique find closed loop transfer function of the system whose block diagram is shown in figure.

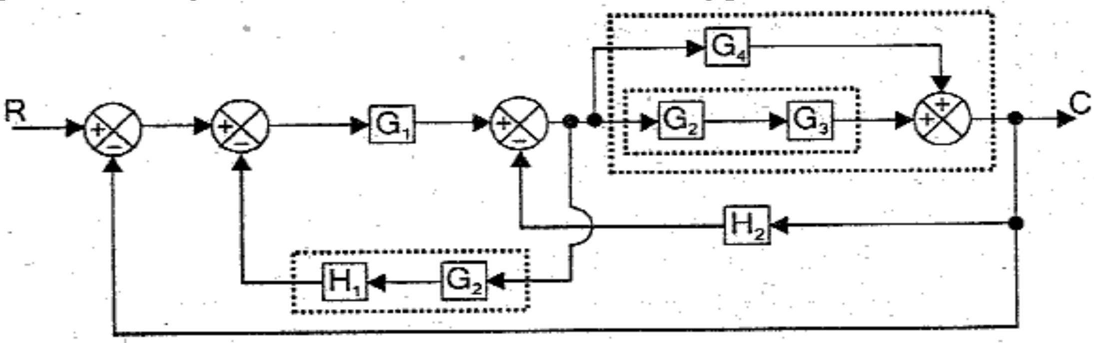


SOLUTION:

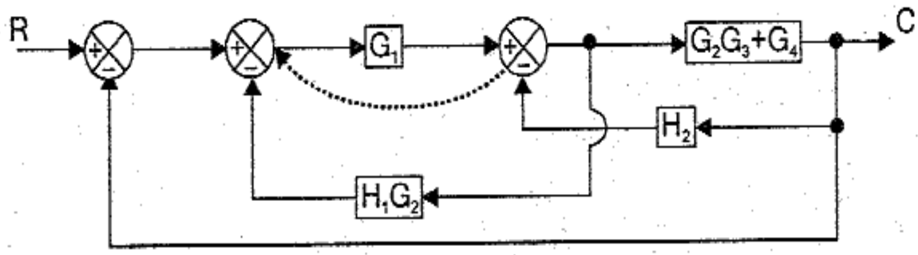
Step 1: Moving the branch point before the block



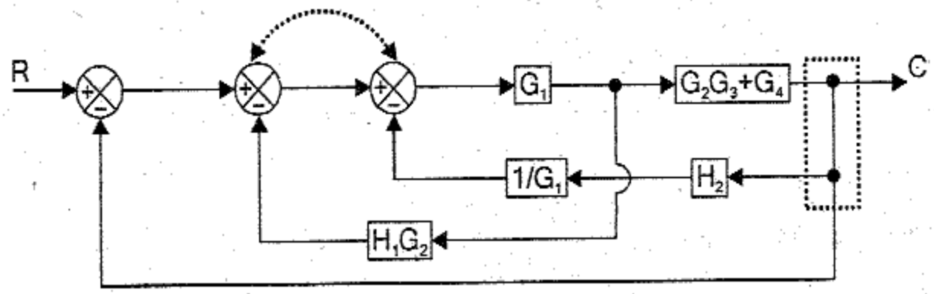
Step2: Combining the blocks in cascade and eliminating parallel blocks



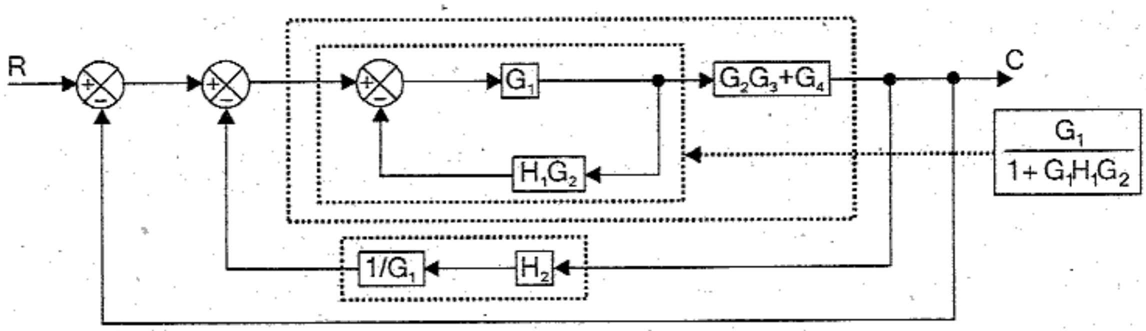
Step 3: Moving summing point before the block.



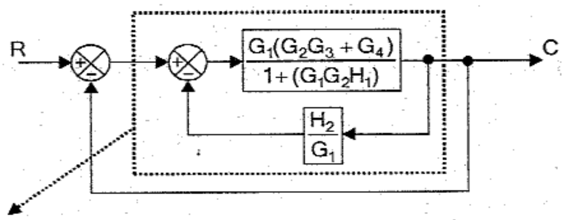
Step4: Interchanging summing points and modifying branch points



Step 5: Eliminating the feedback path and combining blocks in cascade

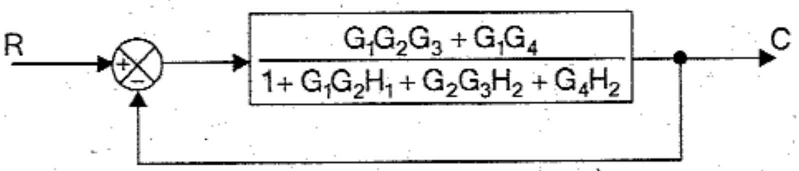


Step6: Eliminating the feedback path



$$\frac{\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}\,\frac{H_2}{G_1}} \Rightarrow \frac{\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1}}{\frac{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}{1+G_1G_2H_1}} \Rightarrow \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}$$

Step 7: Eliminating the feedback path



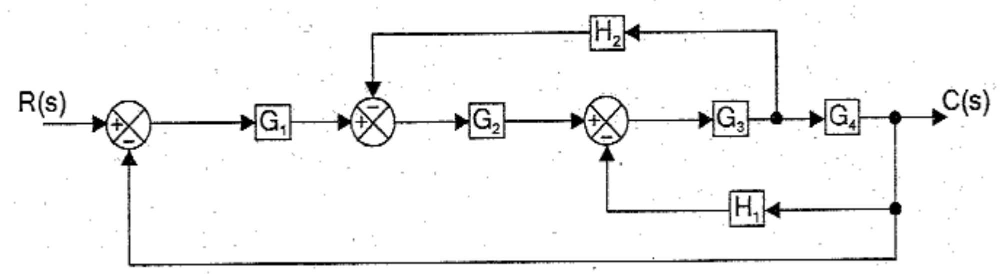
$$\frac{C}{R}=\frac{\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

RESULT:

The Overall Transfer function of the system, $\frac{C}{R}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$

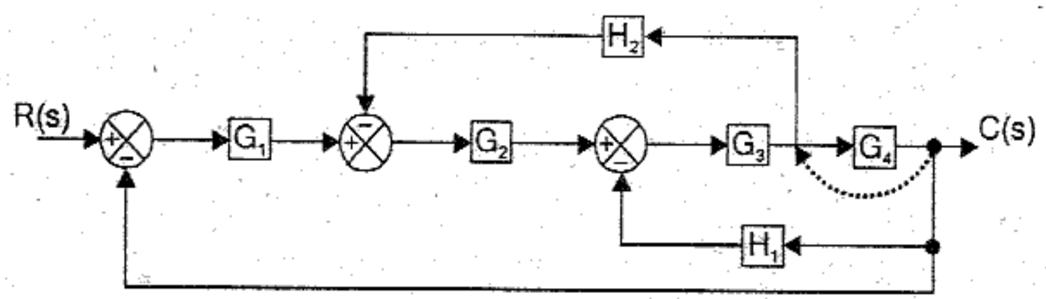
EXAMPLE 3:

Determine the overall Transfer Function C(S)/R(S) for the system shown in figure.

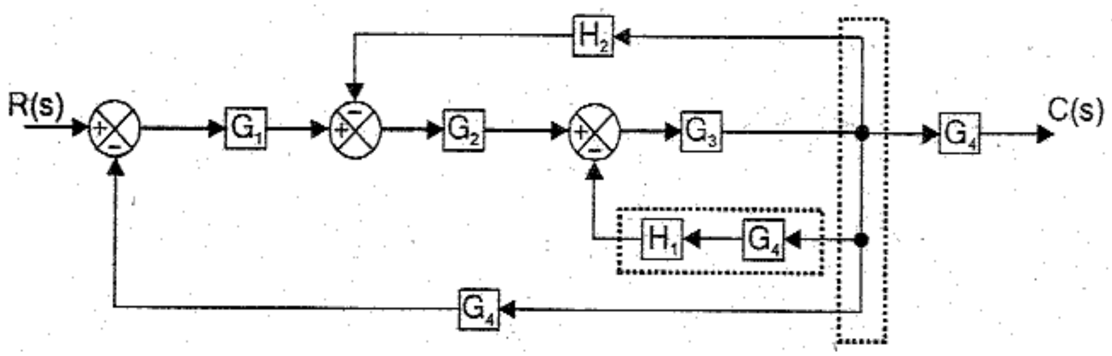


SOLUTION:

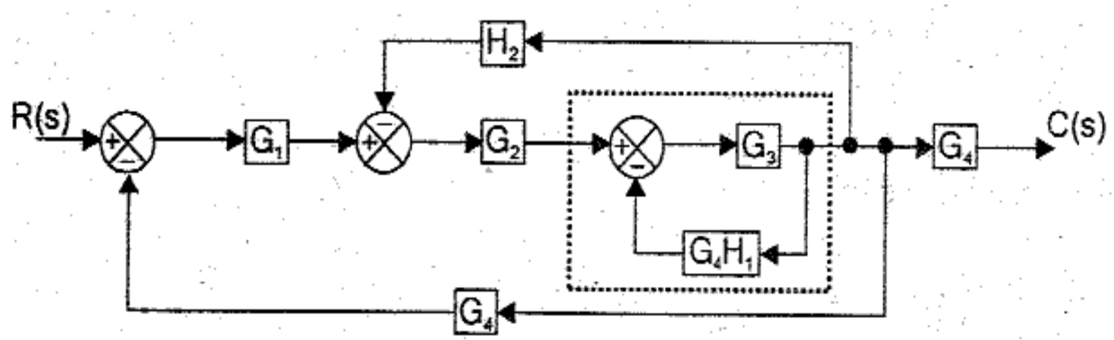
Step 1: Moving the branch point before the block



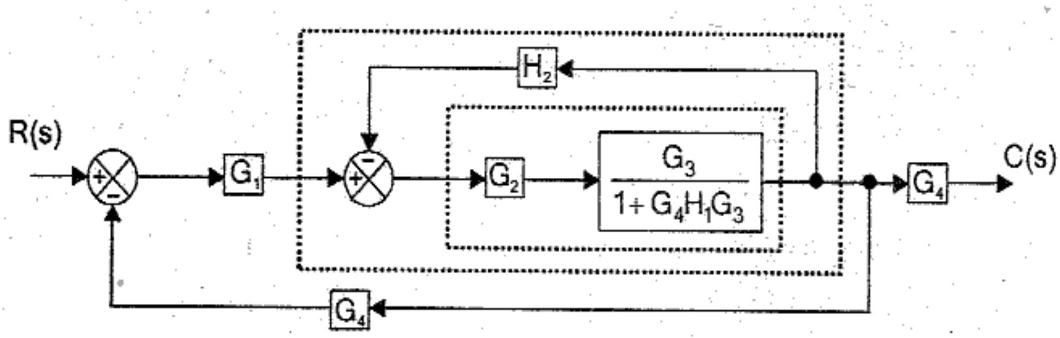
Step2: Combining the blocks In cascade and rearranging the branch points



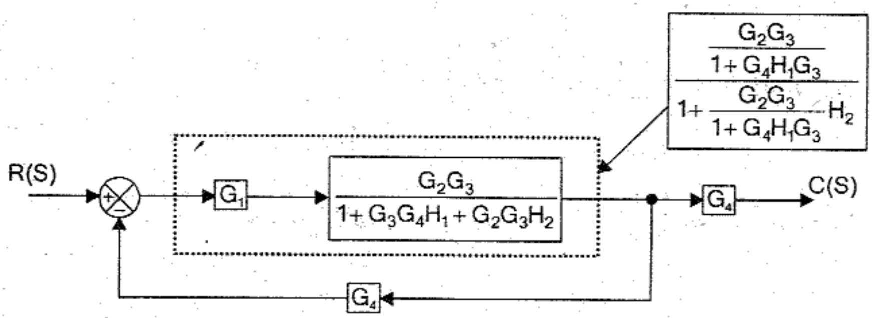
Step 3: Eliminating the feedback path



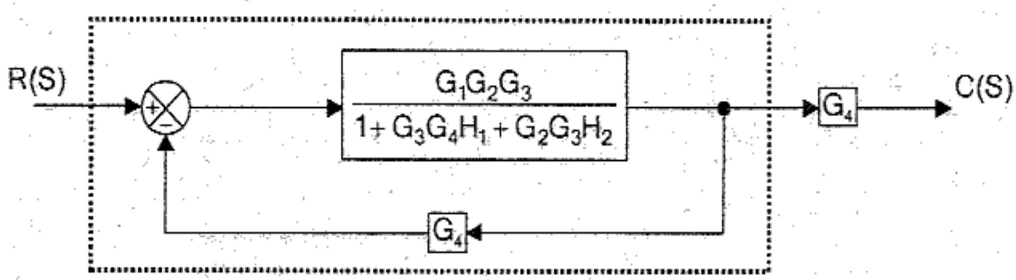
Step4: Combining the blocks in cascade and eliminating feedback path



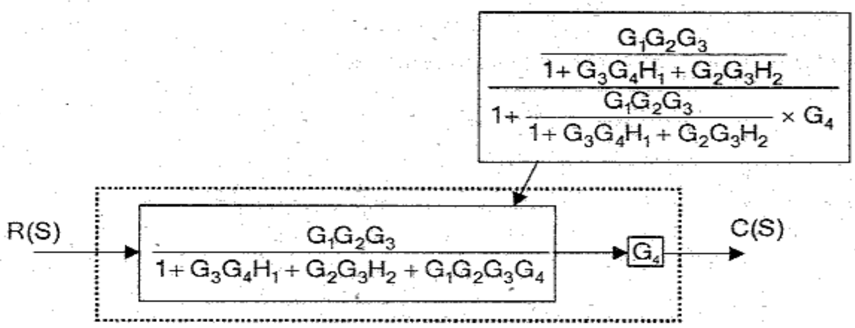
Step 5: Combining the blocks in cascade



Step6: Eliminating the Feedback path



Step 7: Combining the blocks in cascade



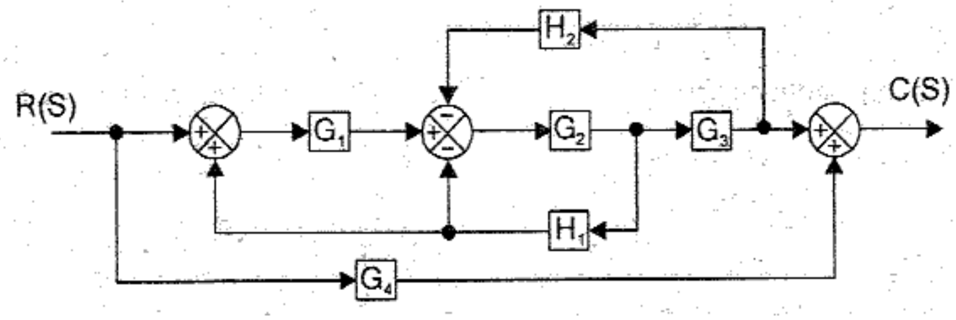
$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

RESULT:

The Overall Transfer function of the system, $\frac{C}{R} = \frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$

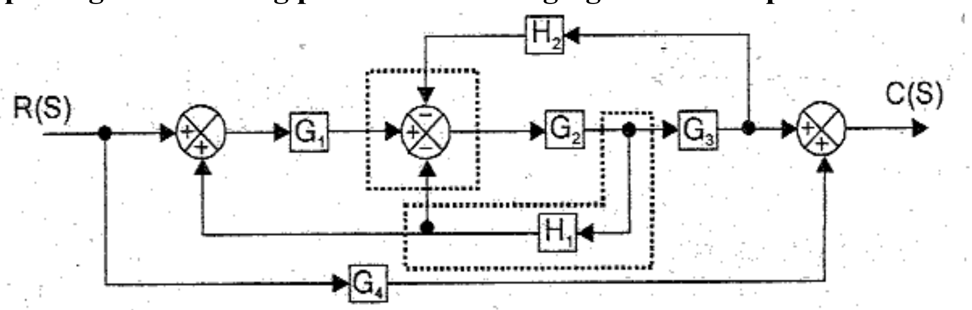
EXAMPLE 4:

Obtain the closed loop transfer function C(s)IR(s) of the system whose diagram is shown in fig.

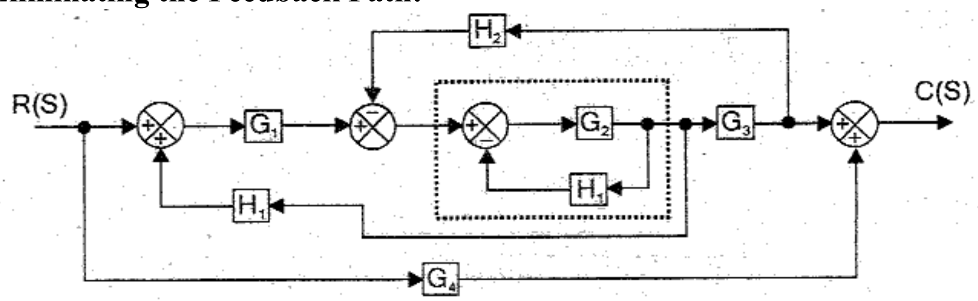


SOLUTION:

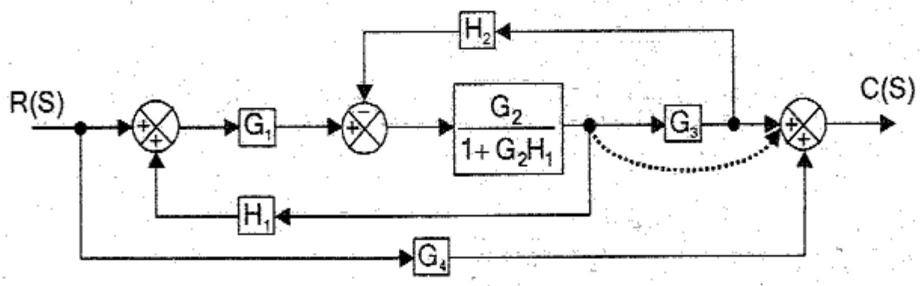
Step 1: Splitting the summing point and rearranging the branch points



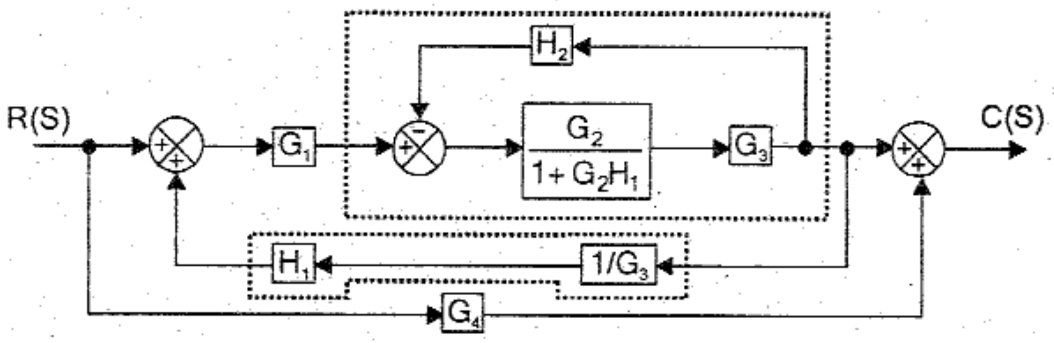
Step2: Eliminating the Feedback Path:



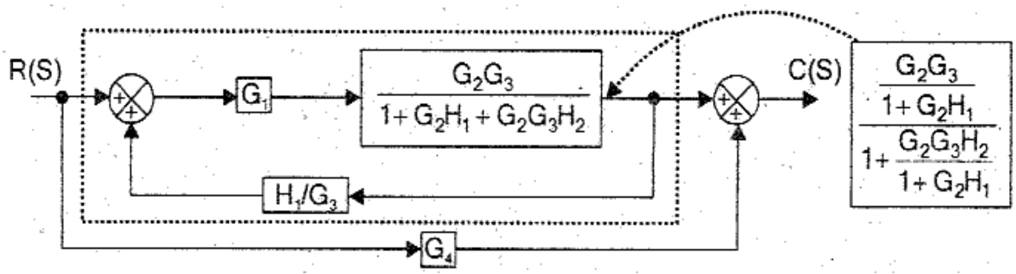
Step 3: Shifting the branch point after the block.



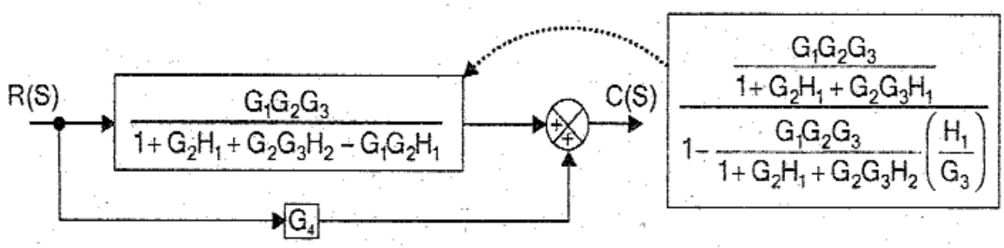
Step4: Combing the blocks in cascade and eliminating feedback path



Step 5: Combining the blocks in cascade and eliminating feedback path



Step 6: Eliminating forward path



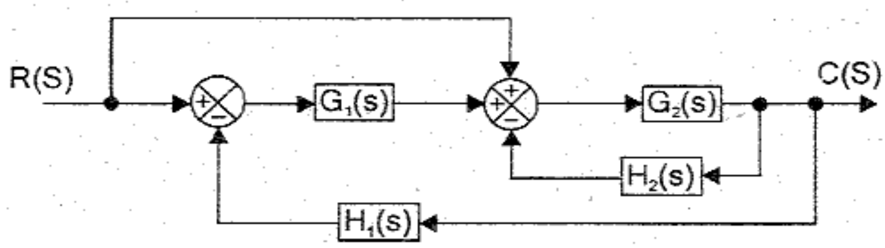
$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$$

RESULT:

The Transfer Function of the system is $\frac{C(S)}{R(S)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$

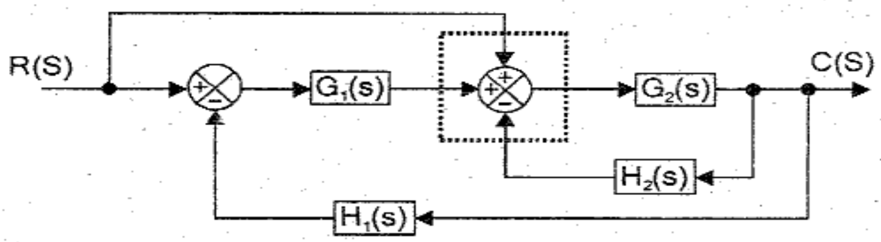
EXAMPLE 5:

The block diagram of a closed loop system is shown in fig. Using the block diagram reduction technique determine the Closed Loop Transfer Function C(s)/R{s).

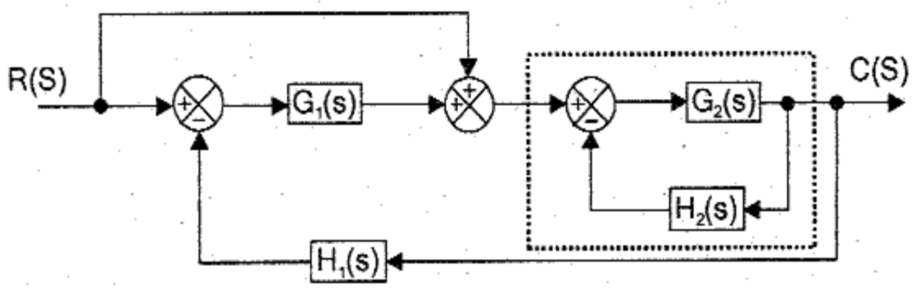


SOLUTION:

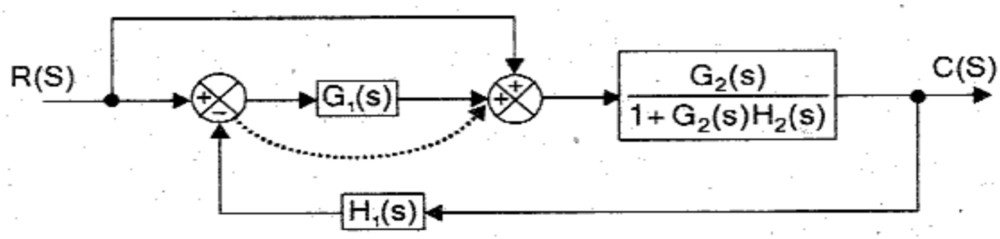
Step1: Splitting the summing point



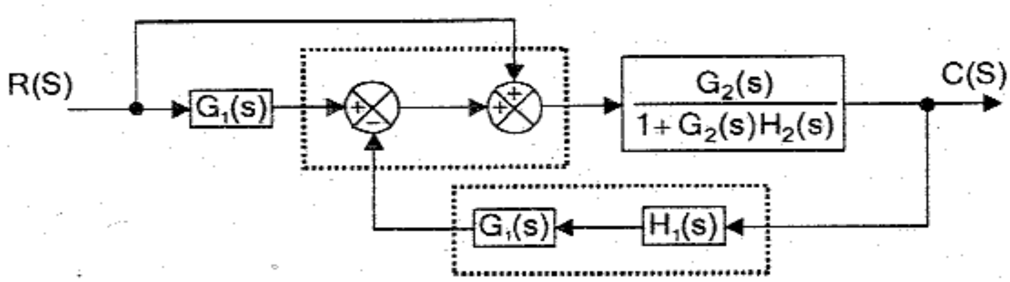
Step 2: Eliminating the feedback path.



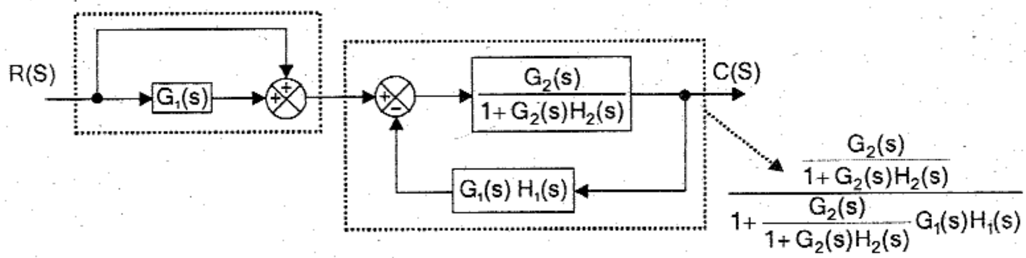
Step 3: Moving the summing point after the block



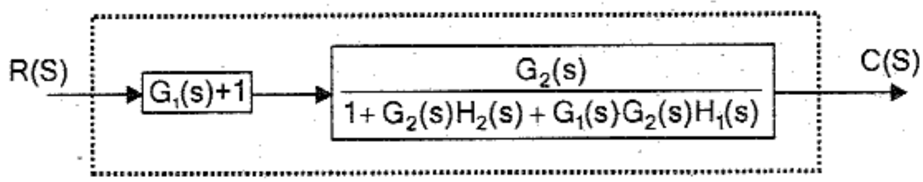
Step4: Interchanging the summing points and combining the blocks in cascade



Step 5: Eliminating the feedback path and feed forward path



Step 6: Combining the blocks in cascade



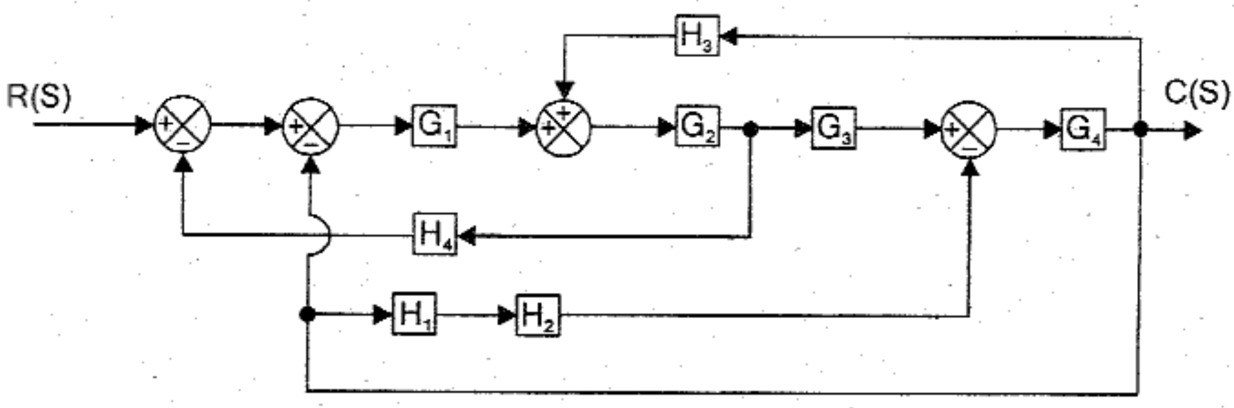
$$\frac{C(s)}{R(s)} = \frac{G_2(s)\,[G_1(s)+1]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

RESULT:

The Transfer Function of the system is $\frac{\mathbf{C(S)}}{\mathbf{R(S)}} = \frac{\mathbf{G_2[G_1+1]}}{\mathbf{1+G_2H_1+G_1G_2H_1}}$

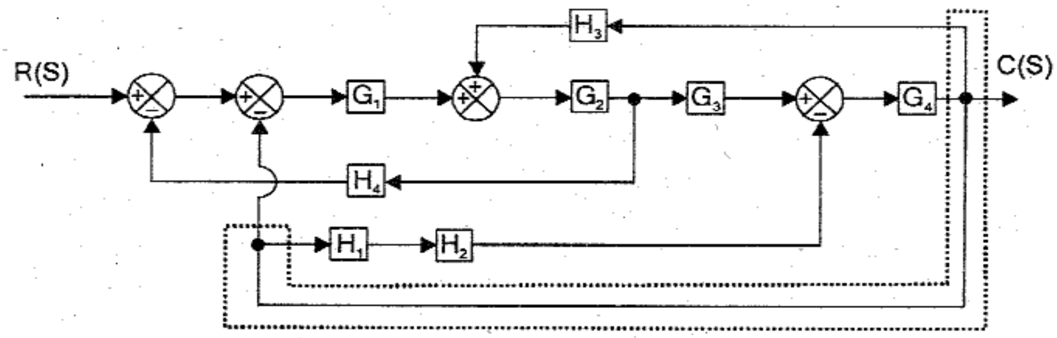
EXAMPLE 6:

Using block diagram reduction technique find the transfer function C(s)/R(s) for the system shown in fig.

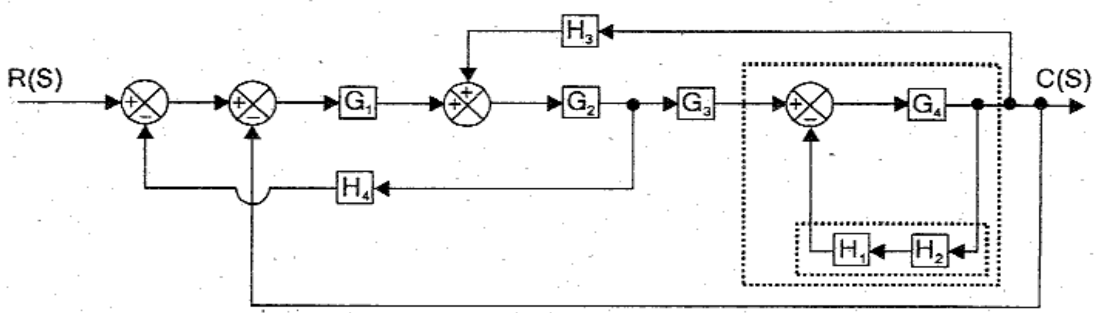


SOLUTION:

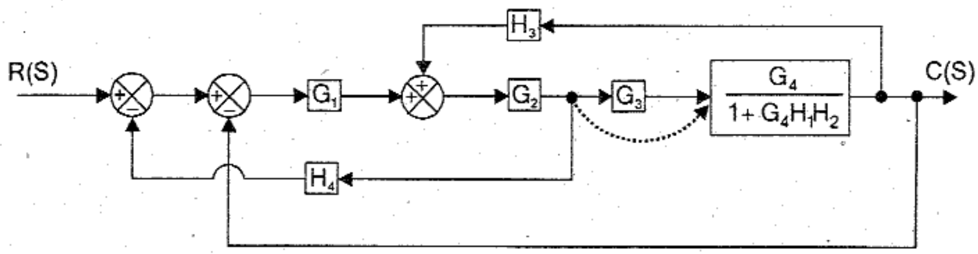
Step 1: Rearranging the Branch points



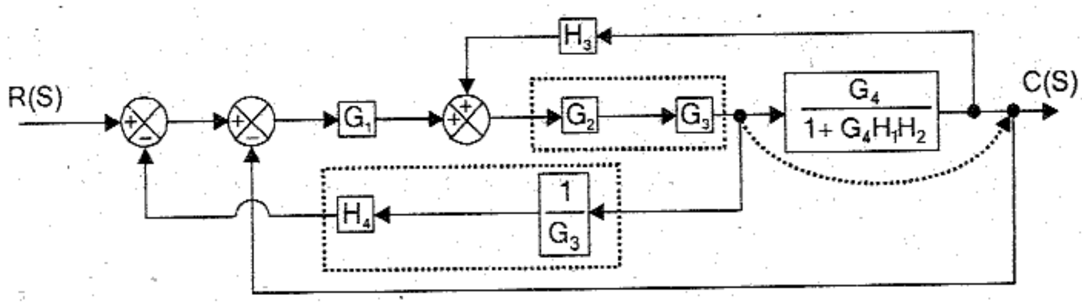
Step 2: Combining the blocks in cascade and eliminating the feedback path.



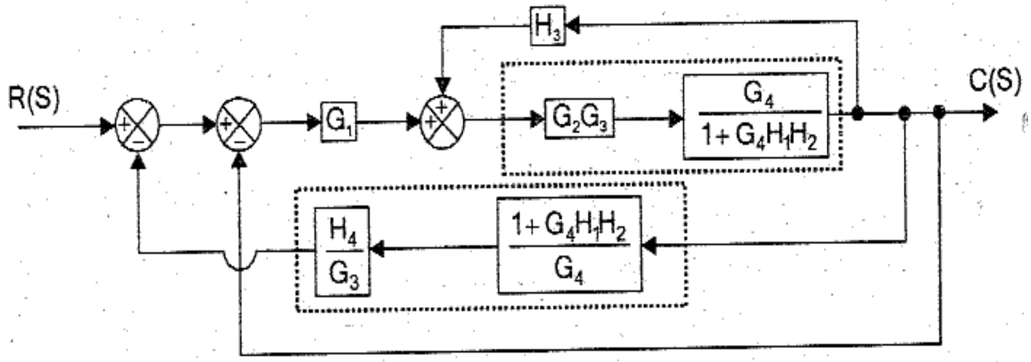
Step 3: Moving the branch point after the block.



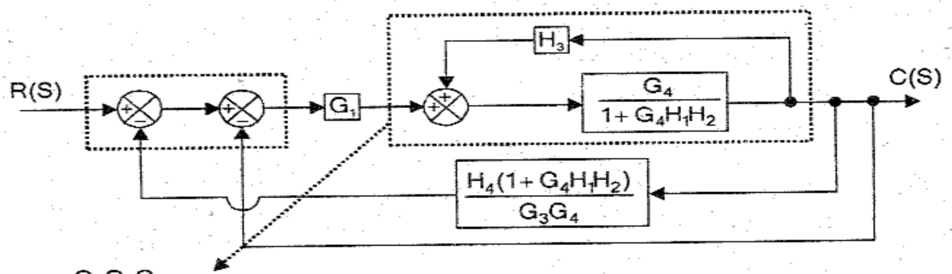
Step 4: Moving the branch point and combining the blocks in cascade



Step 5: Combining the blocks in cascade

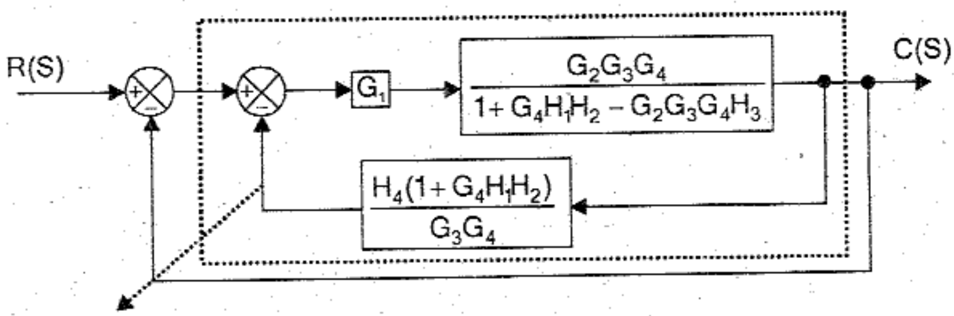


Step 6: Eliminating feedback path and Interchanging the summing points



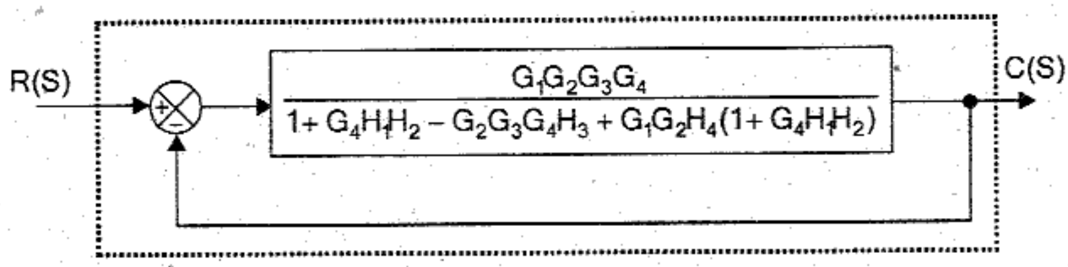
$$\frac{\dfrac{G_2G_3G_4}{1+G_4H_1H_2}}{1-\dfrac{G_2G_3G_4H_3}{1+G_4H_1H_2}} = \frac{G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}$$

Step 7: Combining the blocks in cascade and eliminating the feedback path



$$\frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}}{1+\left(\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}\right)\left(\dfrac{H_4(1+G_4H_1H_2)}{G_3G_4}\right)} = \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}$$

Step 8: Eliminating the unity feedback path



$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}{1+\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}$$

$$= \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)+G_1G_2G_3G_4}$$

$$= \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

RESULT:

The Transfer Function of the system is $\frac{C(S)}{R(S)} = \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_{4H_4})+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$

2.4 SIGNAL FLOW GRAPH:

The signal flow graph is used to represent the control system graphically and it was developed by S.J. Mason. A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The. signal flow graph of the system can be constructed using these equations.

2.4.1 EXPLANATION OF TERMS USED IN SIGNAL FLOW GRAPH:

Node: A node is a point representing a variable or signal.

Branch: A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.

Transmittance: The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real or complex.

Input node (Source): It is a node that has only outgoing branches.

Output node (Sink): It is a node that has only incoming branches.

Mixed node: It is a node that has both incoming and outgoing branches.

Path: A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once.

Open path: A open path starts at a node and ends at another node.

Closed path: Closed path starts and ends at same node.

Forward path: It is a path from an input node to an output node that does not cross any more than once.

Forward path gain: It is the product of the branch transmittances (gains) of a forward path.

Individual loop: It is a closed path starting from a node and after passing through a certain part of a graph arrives at the same node without crossing any node more than once.

Loop gain: It is the product of the branch transmittances (gains) of a loop.

Non-touching Loops: If the loops do not have a common node then they are said to be non-touching loops.

2.4.2 PROPERTIES OF SIGNAL FLOW GRAPH:

The basic properties of signal flow graph are the following:

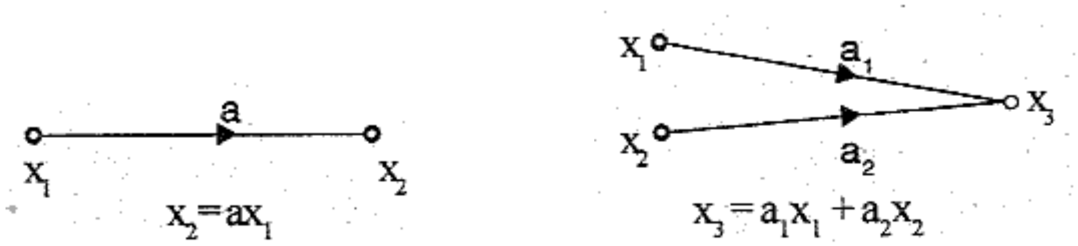
- The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.
- Signal flow graph is applicable to linear systems only.
- A node in the signal flow graph represents the variable or signal.
- A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.
- A mixed node which has both incoming and outgoing signals can be treated as an output node by adding an outgoing branch of unity transmittance.
- A branch indicates functional dependence of one signal on the other.
- The signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.
- The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

2.4.3 RULES FOR SIGNAL FLOW GRAPH:

Signal flow graph for a system can be reduced to obtain the transfer function of the system using the following rules. The guideline in developing the rules for signal flow graph algebra is that the signal at a node is given by sum of all.

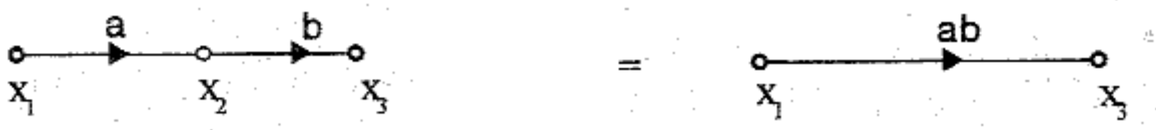
Rule 1: Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch.

Example:



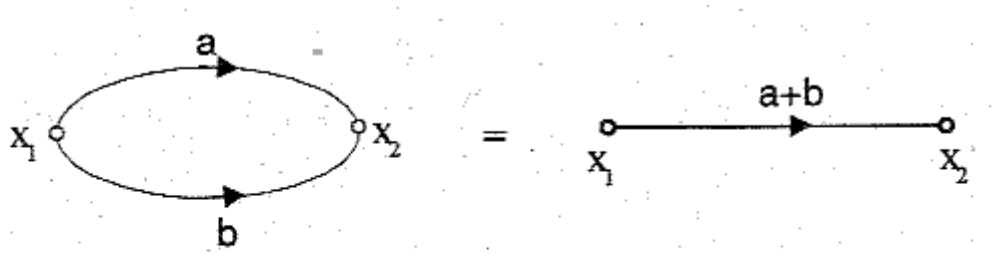
Rule 2: Cascaded branches can be combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Example:



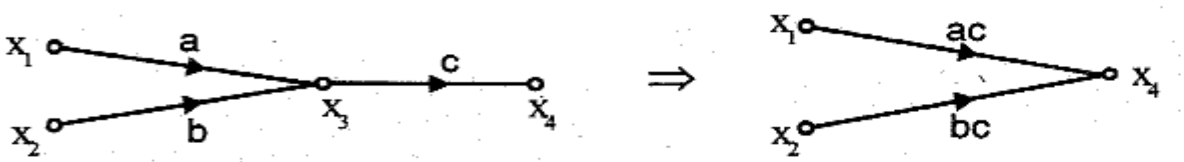
Rule 3: Parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Example:



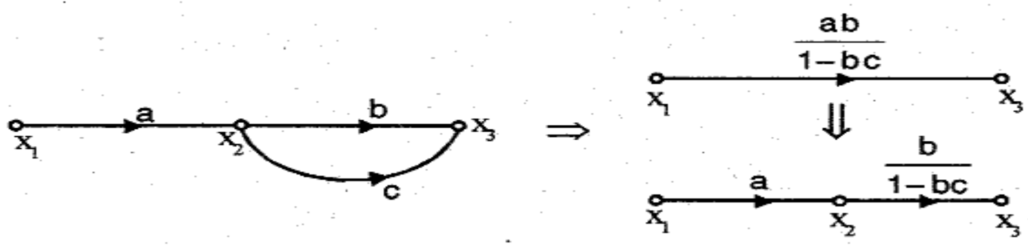
Rule 4: A mixed node can be eliminated by multiplying the transmittance of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

Example:



Rule 5: A loop may be eliminated by writing equations at the input and output node and output node and rearranging the equations to find the ratio of output to input. This ratio gives the gain of resultant branch.

Example:



2.5 SIGNAL FLOW GRAPH REDUCTION:

The signal flow graph of a system can be reduced either by using the rules of a signal flow algebra (i.e.,) by writing equations at every node and then rearranging these equations to get the ratio of output and input (transfer function). The signal flow graph reduction by above method will be time consuming and tedious. S.J.Mason has developed a simple procedure to determine the transfer function of the system represented as a Signal flow graph.

2.5.1 MASON'S GAIN FORMULA:

The Mason's gain formula is used to determine the transfer function of the system from the flow graph of the system.

Let, R(s) = Input to the system

C(s) = Output of the system

$$\boxed{\text{Transfer function of the system, } T(s) = \frac{C(S)}{R(S)}}$$

Mason's gain formula states the overall gain of the system [transfer function] as follows,

$$\boxed{\text{Overall gain, } T = \frac{1}{\Delta} \sum_K P_K \Delta_K}$$

T(s) = Transfer function of the system

P_K = Forward path gain of K^{th} forward path

K = Number of forward paths in the signal flow graph

Δ = 1 - (Sum of individual loop gains)

+ (Sum of gain products of all possible combinations of two non - touching loops)

- (Sum of gain products of all possible combinations of three non - touching loops)

+..............

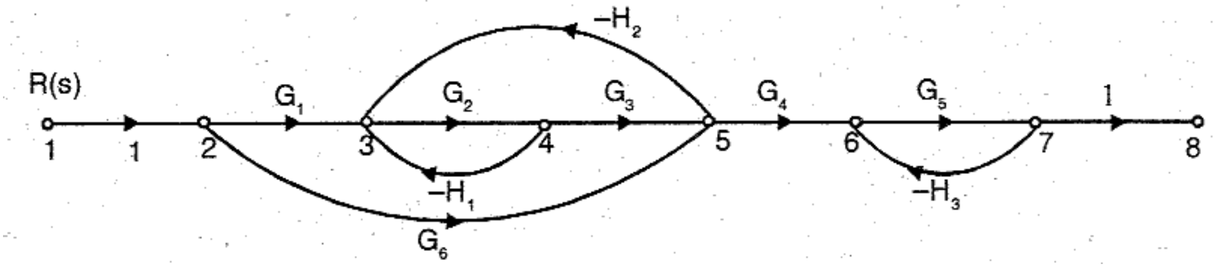
ΔK = Δ for that part of the graph which is not touching K^{th} forward path.

2.5.1 APPLICATIONS OF MASON'S GAIN FORMULAE:

- Easily find out the overall gain of the given control system.
- To determine the input, output relationship, we may use mason's gain formulae.
- It is widely applied to linear system analysis.

EXAMPLE 7:

Find the overall transfer function of the system whose signal flow graph is shown in figure.

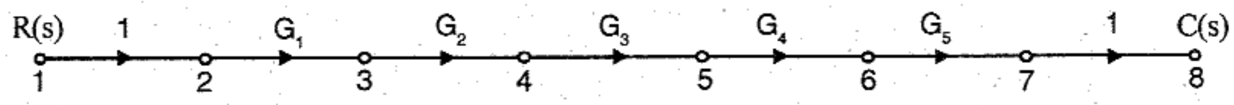


SOLUTION:

I.FORWARD PATH GAINS:

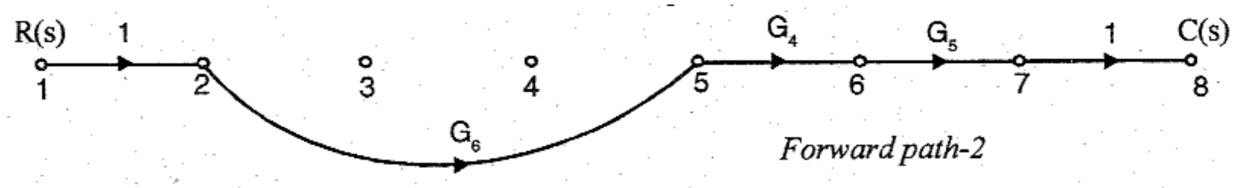
There are two forward path, K =2

Let forward path gains be P1, and P2.



Forward path-1

Gain of forward path-1, $P_1 = G_1\, G_2 G_3 G_4 G_5$

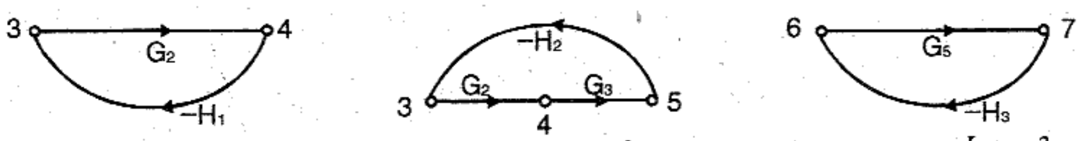


Forward path-2

Gain of forward path-2, $P_2 = G_4\, G_5 G_6$

II.INDIVIDUAL GAIN:

There are three individual loops. let individual loop gains be P_{11} P_{21} and P_{31}.



Loop-1 *Loop-2* *Loop-3*

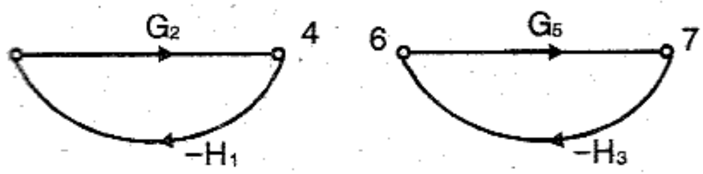
Loop gain of individual loop -1, $P_{11} = -G_2 H_1$

Loop gain of Individual 1oop-2, $P_{21} = -G_2 G_3 H_2$

Loop gain of Individual 1oop-3, $P_{31} = -G_5 H_3$

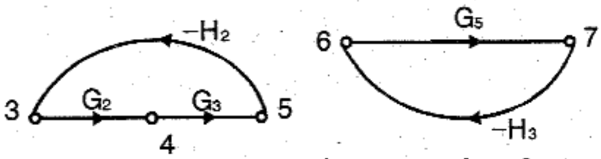
III.GAIN PRODUCTS OF TWO NON-TOUCHING LOOPS:

There are Two combination of 2 non-touching loops. Let the gain products of 2 non-touching loops be P_{12} P_{22}.



First combination of 2 non-touching loops

Gain product of first combination of two non-touching loops $P_{12} = P_{11}P_{31} = (-G_2 H_1)(- G_5H_3) = G_2 G_5 H_1 H_3$.



Second combination of 2 non-touching loops

Gain product of second combination of two non-touching loops $P_{22} = P_{21}P_{31} = (-G_2 G_3 H_2)(-G_5H_3) = G_2 G_3 G_5 H_2 H_3$.

IV. CALCULATION OF Δ AND Δ_K:

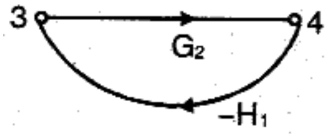
$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$

$= 1 - (-G_2H_1 - G_2G_3H_2 - G_5H_3) + (G_2G_5H_1H_3 + G_2G_3G_5H_2H_3)$

$= 1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3$

$\Delta_1 = 1$., Since there is no part of graph which is not touching with first forward path.

The part of the graph which is non-touching with second forward path is shown in figure.



$\Delta_2 = 1 - (P_{11})$

$\Delta_2 = 1 + G_2H_1$

V. TRANSFER FUNCTION, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is 2 and so K = 2)}$$

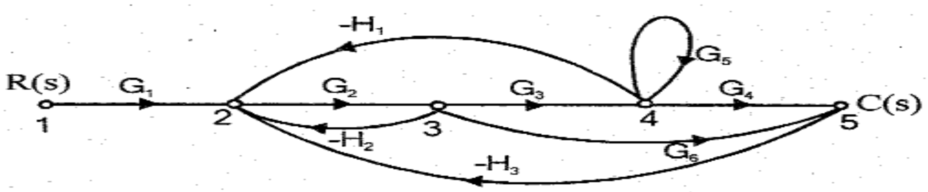
$$T = \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6(1+G_2H_1)}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}$$

RESULT:

$$\boxed{T = \frac{\mathbf{G_1G_2G_3G_4G_5+G_4G_5G_61+G_4G_5G_6G_2H_1}}{\mathbf{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}}}$$

EXAMPLE 8:

Find the overall transfer gain C(S) / R(S) for the signal flow graph is shown in figure.

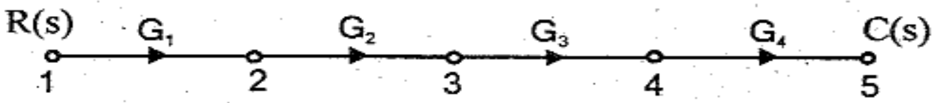


SOLUTION:

I.FORWARD PATH GAINS:

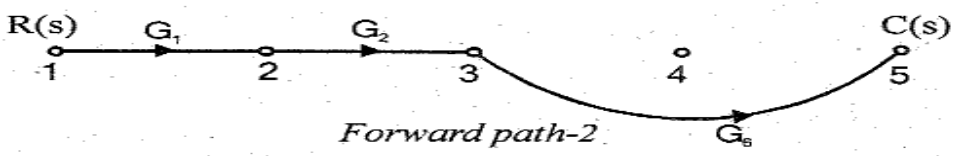
There are two forward path, K =2

Let forward path gains be P1, and P2.



Forward path-1

Gain of forward path-1, $P_1 = G_1\,G_2G_3G_4$

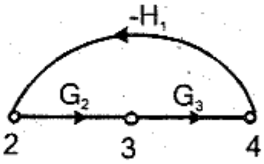


Forward path-2

Gain of forward path-2, $P_2 = G_1\, G_2 G_6$

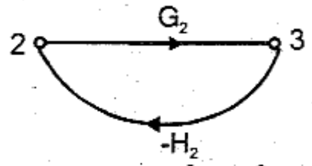
II.INDIVIDUAL GAIN:

There are five individual loops. Let individual loop gains be $P_{11,}$ $P_{21,}$ $P_{31}$and $_{41}$ P_{51}.



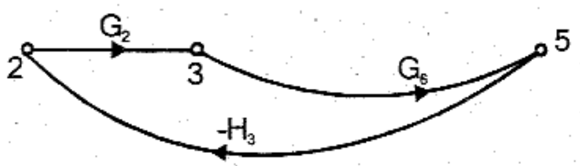
loop-1

Loop gain of individual loop -1, $P_{11} = -G_2\, G_3 H_1$



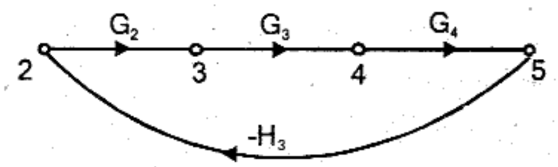
loop-2

Loop gain of Individual 1oop-2, $P_{21} = -G_2\, H_2$



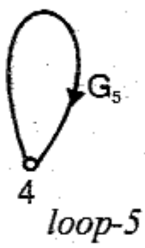
loop-3

Loop gain of Individual 1oop-3, $P_{31} = -G_2\, G_6\, H_3$



loop-4

Loop gain of individual loop -4, $P_{41} = -G_2\, G_3 G_4\, H_3$



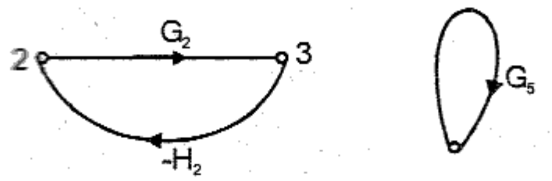
loop-5

Loop gain of Individual 1oop-5, $P_{51} = G_5$

III.GAIN PRODUCTS OF TWO NON-TOUCHING LOOPS:

There are Two combination of 2 non-touching loops.

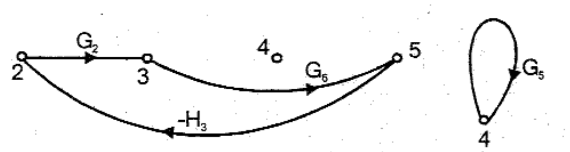
Let the gain products of 2 non-touching loops be P_{12} and P_{22}.



First combination of two non-touching loops

Gain product of first combination of two non-touching loops

$P_{12} = P_{21}P_{51} = (-G_2 H_2)(G_5) = - G_2 G_5 H_2$.



Second combination of two non-touching loops

Gain product of second combination of two non-touching loops

$P_{22} = P_{31}P_{51} = (-G_2 G_6 H_3)(G_5) = - G_2G_5G_6 H_3$.

IV. CALCULATION OF Δ AND Δ_K:

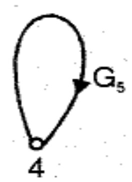
$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51}) + (P_{12} + P_{22})$

$= 1 - (-G_2 G_3H_1 - G_2H_2 - G_2 G_3G_4H_3 + G_5 - G_2G_6H_3) + (- G_2 G_5 H_2 - G_2G_5G_6 H_3)$

$= 1 + G_2 G_3H_1 + G_2H_2 + G_2 G_3G_4H_3 - G_5 + G_2G_6H_3 - G_2 G_5 H_2 - G_2G_5G_6 H_3$

$\Delta_1 = 1$., Since there is no part of graph which is not touching with first forward path.

The part of the graph which is non-touching with second forward path is shown in figure.



$\Delta_2 = 1 - G_5$

V. TRANSFER FUNCTION, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_{K} P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is 2 and so K = 2)}$$

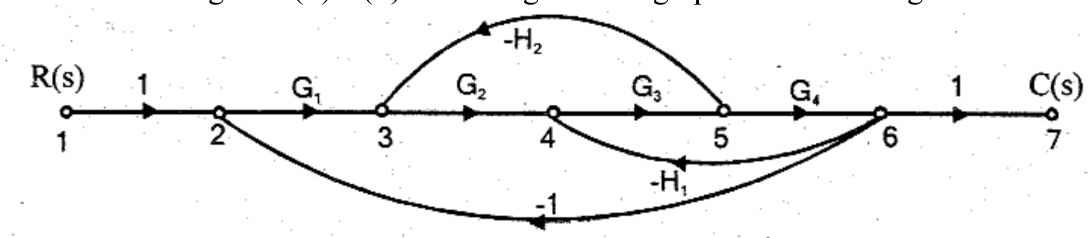
$$T = \frac{G_1G_2G_3G_4(1)+G_1G_2G_6(1-G_5)}{1+G_2G_3H_1+G_2H_2+G_2\,G_3G_4H_3-\,G_5-G_2G_5H_2+G_2G_6H_3-G_2G_5G_6H_3}$$

RESULT:

$$\mathbf{T = \frac{G_1G_2G_3G_4+G_1G_2G_6-G_1G_2G_5G_6}{1+G_2G_3H_1+G_2H_2+G_2\,G_3G_4H_3-\,G_5-G_2G_5H_2+G_2G_6H_3-G_2G_5G_6H_3}}$$

EXAMPLE 9:

Find the overall gain C(S)/R(S) for the signal flow graph is shown in figure.

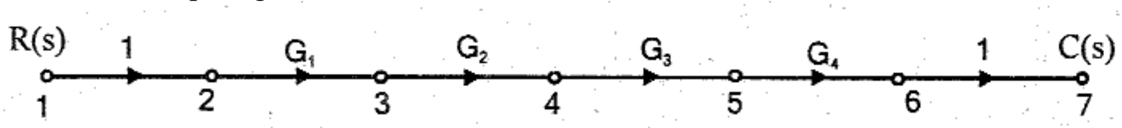


SOLUTION:

I.FORWARD PATH GAINS:

There is only one forward path, K =1

Let the forward path gain be P1.

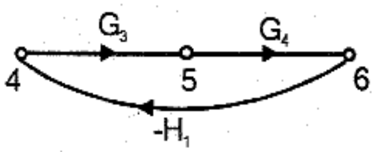


Forward path-1

Gain of forward path-1, $P_1 = G_1\,G_2G_3G_4$

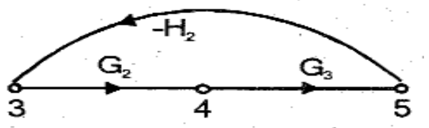
II.INDIVIDUAL GAIN:

There are three individual loops. let individual loop gains be P_{11} P_{21} and P_{31}.



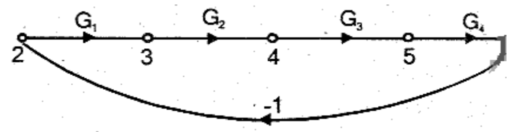
loop-1

Loop gain of individual loop -1, $P_{11} = -G_3\,G_4\,H_1$



loop-2

Loop gain of Individual 1oop-2, $P_{21} = -G_2\, G_3\, H_2$



loop-3

Loop gain of Individual 1oop-3, $P_{31} = -G_1\, G_2\, G_3\, G_4$

III.GAIN PRODUCTS OF TWO NON-TOUCHING LOOPS:

There is no possible combination of two non-touching loops, three non-touching loops, etc.,

IV. CALCULATION OF Δ AND Δ_K:

$\Delta = 1 - (P_{11} + P_{21} + P_{31})$

$= 1 - (-G_3\, G_4\, H_1 - G_2\, G_3\, H_2 - G_1\, G_2\, G_3\, G_4)$

$= 1 + G_3\, G_4\, H_1 + G_2\, G_3\, H_2 + G_1\, G_2\, G_3\, G_4$

$\Delta_1 = 1$., Since there is no part of graph which is not touching with first forward path.

V. TRANSFER FUNCTION, T

By Mason's gain formula the transfer function, T is given by,

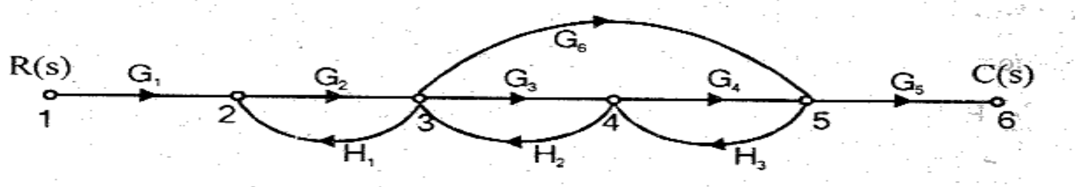
$$T = \frac{C(s)}{R(s)} = \frac{1}{\Delta} \sum_K P_K \Delta_K = \frac{1}{\Delta} P_1 \Delta_1 \text{ (Number of forward path is 1 and so K = 1)}$$

RESULT:

$$\mathbf{T} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4}$$

EXAMPLE 10:

The Signal Flow Graph for a feedback system is shown in figure. Determine the closed loop Transfer Function.

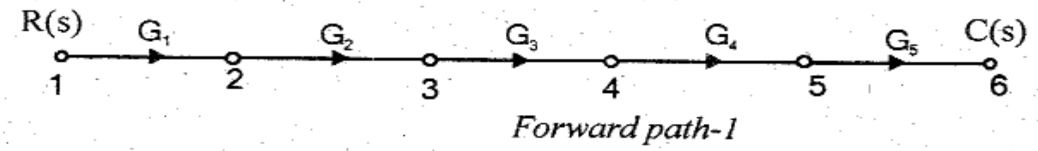


SOLUTION:

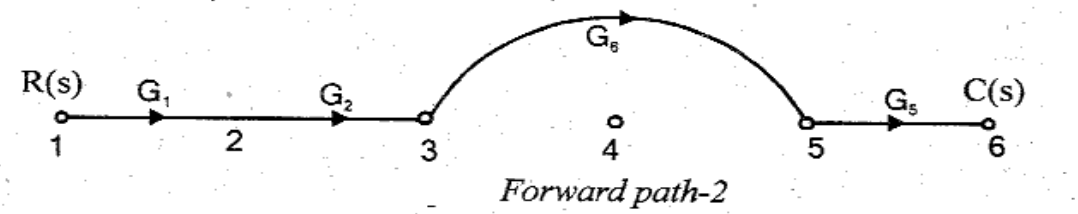
I.FORWARD PATH GAINS:

There are two forward path, K =2

Let forward path gains be P1, and P2.



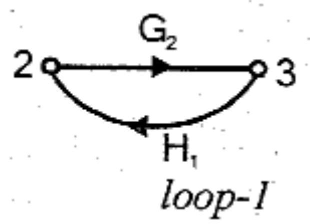
Gain of forward path-1, $P_1 = G_1\ G_2G_3G_4G_5$



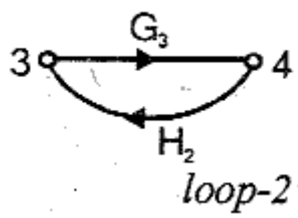
Gain of forward path-2, $P_2 = G_1\ G_2G_5\ G_6$

II.INDIVIDUAL GAIN:

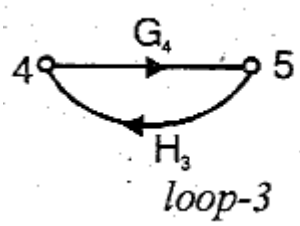
There are Four individual loops. Let the individual loop gains be P_{11}, P_{21}, P_{31} and P_{41}.



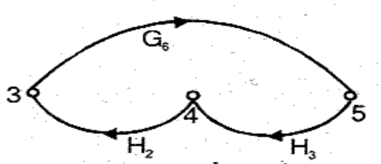
Loop gain of individual loop -1, $P_{11} = G_2\ H_1$



Loop gain of Individual 1oop-2, $P_{21} = G_3\ H_2$



Loop gain of Individual 1oop-3, $P_{31} = G_4\ H_3$

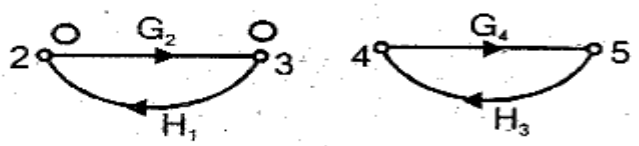


loop-4

Loop gain of Individual 1oop-3, $P_{41} = G_6H_2\,H_3$

III.GAIN PRODUCTS OF TWO NON-TOUCHING LOOPS:

There is only one combination of two non-touching loops. Let the gain products of two non-touching loops be P_{12}



First combination of two non touching loops

Gain product of first combination of two non-touching loops

$P_{12} = P_{11}P_{31} = (G_2\,H_1)(G_4H_3) = G_2\,G_4\,H_1\,H_3.$

IV. CALCULATION OF Δ AND Δ_K:

$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12})$

$= 1 - (G_2\,H_1 + G_3\,H_2 + G_4\,H_3 + G_6H_2\,H_3) + (G_2\,G_4\,H_1\,H_3)$

$= 1 - G_2\,H_1 - G_3\,H_2 - G_4\,H_3 - G_6H_2\,H_3 + G_2\,G_4\,H_1\,H_3$

Since there is no part of graph which is not touching with first forward path-1, path-2,
$\Delta_1 = \Delta_2 = 1$.

V. TRANSFER FUNCTION, T

By Mason's gain formula the transfer function, T is given by,

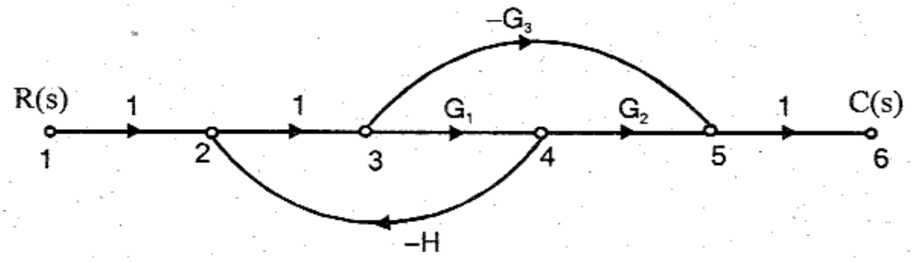
$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is 2 and so K = 2)}$$

RESULT:

$$\boxed{T = \frac{G_1G_2G_3G_4G_5 + G_1G_2G_5G_6}{1 - G_2H_1 - G_3H_2 - G_4H_3 - G_6H_2H_3 + G_2G_4H_1H_3}}$$

EXAMPLE 10:

Find the overall transfer function of the system whose signal flow graph is shown in figure.

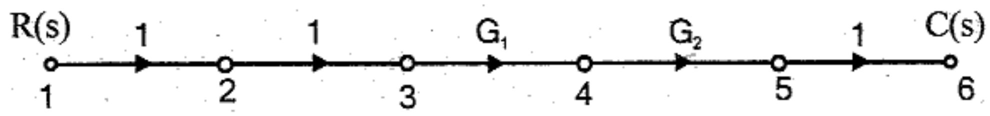


SOLUTION:

I.FORWARD PATH GAINS:

There are two forward path, K =2

Let forward path gains be P1, and P2.



Forward path-1

Gain of forward path-1, $P_1 = G_1\ G_2$

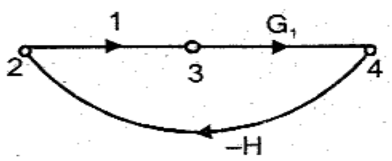
Forward path-2

Gain of forward path-2, $P_2 = -\ G_3$

II.INDIVIDUAL GAIN:

There is only one individual loop. Let the individual loop gain be P_{11}.

Loop gain of individual loop -1, $P_{11} = -G_1\ H$



loop-1

III.GAIN PRODUCTS OF TWO NON-TOUCHING LOOPS:

There is no combination of two non-touching loops.

IV. CALCULATION OF Δ AND Δ_K:

$$\Delta = 1 - (P_{11})$$
$$= 1 - (-G_1H)$$
$$=1+G_1H$$

Since there is no part of graph which is not touching with first forward path-1 and path-2, $\Delta 1=\Delta 2=1$.

V. TRANSFER FUNCTION, T

By Mason's gain formula the transfer function, T is given by,

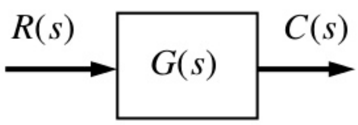
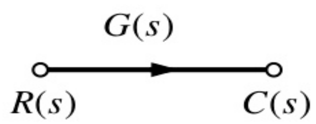
$$T = \frac{1}{\Delta}\sum_K P_K \Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \quad \text{(Number of forward paths is 2 and so K = 2)}$$

$$T = \frac{G_1G_2(1) - G_3\,(1)}{1+G_1H}$$

RESULT:

$$\boxed{T = \frac{G_1G_2 - G_3}{1+G_1H}}$$

2.6 COMPARISON OF BLOCK DIAGRAM REDUCTION AND SIGNAL FLOW GRAPH:

BLOCK DIAGRAM	SIGNAL FLOW GRAPH
block diagram:  • In this case at each step Block diagram is to be redrawn. That's why it is Tedious method. So, wastage of time and space.	**signal flow graph:**  • Only one time SFG is to be drawn and the Mason's gain formulae is to be evaluated. So, time and space is saved.

PART – A:

1.What are the Basic components of Block diagram?

2.What is Block diagram?

3.List any two advantages of Block diagram.

4.Write any two disadvantages of block diagram.

5.What is Block diagram reduction technique?

6.What is signal flow graph?

7.What is Node?

8.What is Branch?

9.Define Transmittance.

10.What is Input Node (or) Source?

11.What is Output node (or) Sink?

12.Define mixed node.

13.What is Path?

14.Define Open path.

15.What is closed path?

16.What is Forward path?

17.What is Forward path gain?

18.Define Loop gain.

19.Define Non-Touching Loop.

20.Write the applications of Mason's gain formulae

PART – B:

1.Write the rule for Combine the blocks in Cascade.

2. Write the rule for Combine the blocks in parallel.

3. Write the rule for Move the branch point ahead of the block.

4. Write the rule for Move the branch point before the block.

5. Write the rule for Move the summing point ahead of the block.

6. Write the rule for Move the summing point before the block.

7.Why is negative feedback mainly used in closed loop system?

8.Reduce the Negative feedback loop.

9.State the rule for Eliminating the positive feedback loop.

10.Write about Mason's gain formulae.

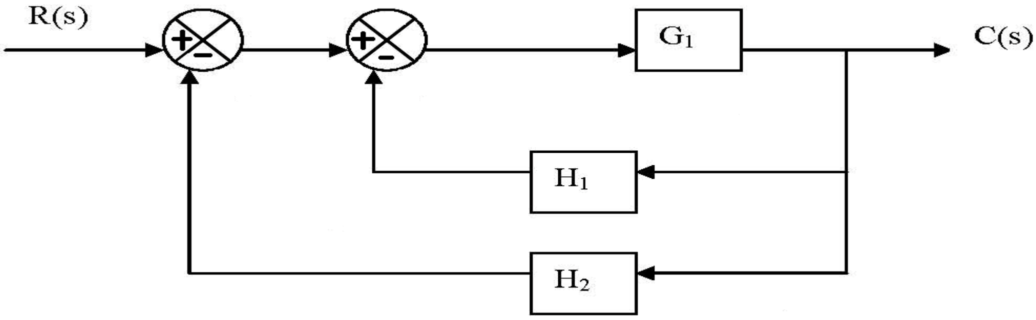
11.Define Individual loop.

12.Define any two properties of Signal flow graph.

13.Compare the difference between Block diagram reduction & Signal flow graph.

PART – C:

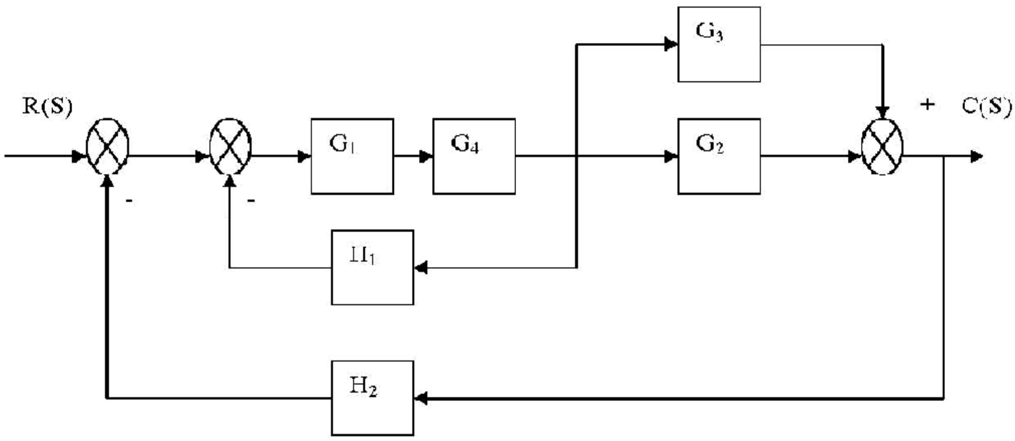
1. Using block diagram reduction technique find the transfer function C(s)IR(s) for the system shown in fig.



Ans:

$$\frac{C(S)}{R(S)} = \frac{G_1}{1+G_1H_1+G_1H_2}$$

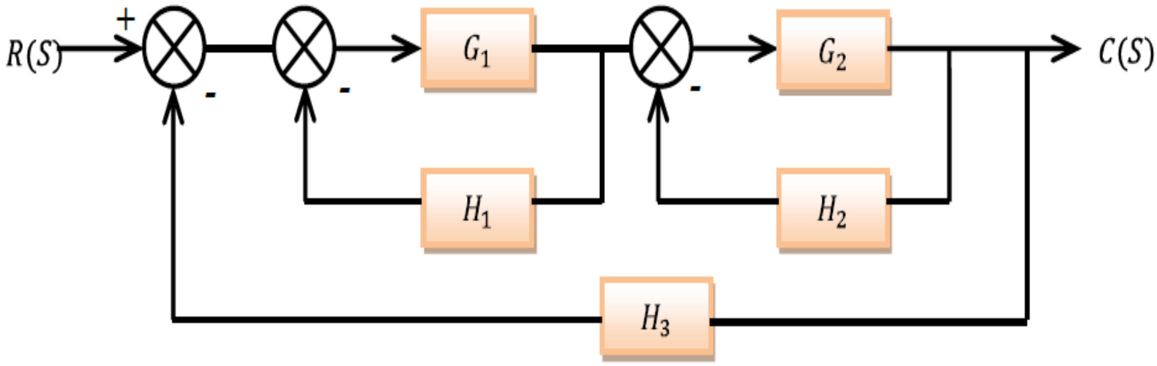
2. The block diagram of a closed loop system is shown in fig. Using the block diagram reduction technique determine the Closed Loop Transfer Function C(s)/R{s).



Ans :

$$\frac{C(S)}{R(S)} = \frac{G_1G_2G_4+G_1G_3G_4}{1+G_1G_4H_1+G_1G_2G_4+G_1G_3G_4}$$

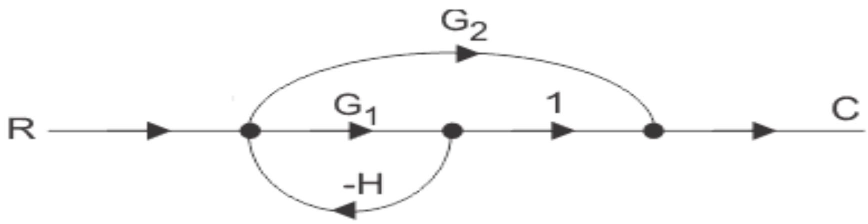
3. The block diagram of a closed loop system is shown in fig. Using the block diagram reduction technique determine the Closed Loop Transfer Function C(s)/R{s).



Ans:

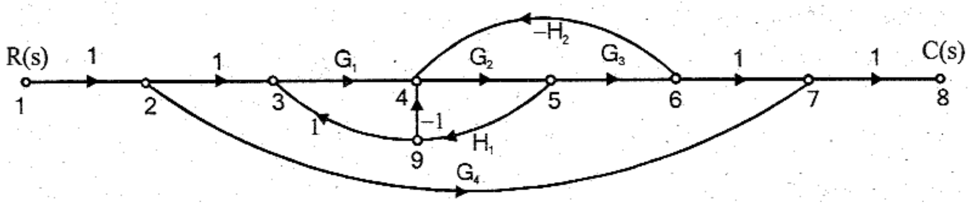
$$TF = \frac{C(S)}{R(S)} = \frac{G_1 G_2}{(1 + G_1 H_1)(1 + G_2 H_2) + G_1 G_2 H_3}$$

4. Find the overall gain C(S)/R(S) for the signal flow graph is shown in figure.



$$T = \frac{C}{R} = \sum_{k=1}^{2} \frac{P_k \Delta_k}{\Delta} = \frac{P_1 \cdot \Delta_1 + P_2 \cdot \Delta_2}{\Delta} = \frac{G_1 + G_2}{1 + G_1 H}$$

5. The Signal Flow Graph for a feedback system is shown in figure. Determine the closed loop Transfer Function.



Ans:

$$T = \frac{G_1G_2G_3 + G_4 + G_1G_2G_4H_1 + G_2G_3G_4H_2 + G_2G_4H_1}{1 + G_1G_2H_1 + G_2G_3H_2 + G_2H_1}$$

UNIT III

TIME RESPONSE ANALYSIS

3.1 TIME RESPONSE

Time Response is the output of the closed loop system c(t) as a function of time. Time Response can be obtained when the transfer function and the input to the system are known. It is denoted as c(t). It is given by Inverse Laplace transform of the product of the input and transfer function of the system.

Response in s-Domain

Transfer function = $\frac{C(s)}{R(s)}$

$$C(s) = \frac{\mathbf{G(s)}}{\mathbf{1+G(s)H(s)}} \times R(s) \qquad \text{...... (3.1)}$$

Taking Inverse Laplace transform on both sides

Response in t-domain

$$c(t) = \mathcal{L}^{-1}\,[C(s)] = \quad \mathcal{L}^{-1}\,\frac{G(s)R(s)}{1+G(s)H(s)} \qquad \text{....... (3.2)}$$

Time Response of a control system consists of two parts:

1. Transient response
2. Steady state response.

Transient response:

1. It shows the response of the system when the input changes from one state to another state.
2. Transient response is dependent upon the system poles only and not on the type of input.

Steady state response:

1. It shows the response of the system as time t approaches infinity.
2. The steady-state response depends on the system dynamics and the input quantity.

3.2 STANDARD TEST SIGNALS

To predict the response of the system we require input signal. **The characteristics of input signal are a sudden shock, a sudden change, a constant velocity and a constant acceleration.** Hence test signals which resemble these characteristics are used as input signals to predict the performance of the system.

The commonly used test signals are

i. Step signal
ii. Ramp signal
iii. Parabolic signal
iv. Sinusoidal signal
v. Impulse signal

3.2.1 STEP SIGNAL

The step signal is a signal whose value changes from zero to A at t=0 and remains constant at A for t > 0. The step signal resembles an actual steady input to a system.

A special case of step signal is unit step in which A is unity.

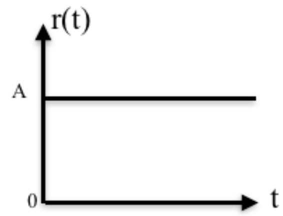


Fig 3.1 : Step Signal

The mathematical representation of the step signal is

$$r(t) = A;\ t \geq 0$$
$$\quad = 0;\ t < 0$$

$$R(s) = \frac{A}{s} \qquad \ldots\ldots (3.3)$$

For unit step signal r(t) = 1, $R(s) = \frac{1}{s}$

3.2.2 RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time. The ramp signal resembles a constant velocity input to the system. Ramp signal is the integral of step signal. A special case of ramp signal is unit ramp signal in which the value of A is unity.

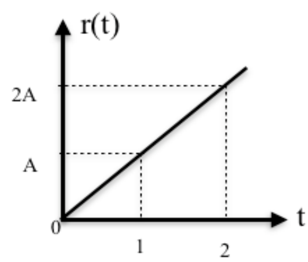


Fig 3.2 : Ramp Signal

The mathematical representation of the ramp signal is

$r(t) = At;\ t \geq 0$

$\quad = \ 0;\ t < 0$

$$R(s) = \frac{A}{s^2} \qquad \ldots\ldots (3.4)$$

For unit ramp signal $r(t) = t$, $R(s)= \frac{1}{s^2}$

3.2.3 PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t = 0. The parabolic signal resembles a constant acceleration input to the system. Parabolic signal is an integral of ramp signal. A special case of parabolic signal is unit parabolic signal in which A is unity.

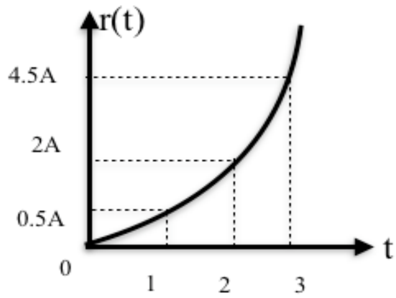


Fig 3.3 : Parabolic Signal

The mathematical representation of the parabolic signal is,

$r(t) = A\frac{t^2}{2};\ t \geq 0$

$\quad = 0;\ t < 0$

$$R(s) = \frac{A}{s^3} \qquad \ldots\ldots (3.5)$$

For unit parabolic signal $r(t) = \frac{t^2}{2}$, $R(s)= \frac{1}{s^3}$

3.3 ORDER OF THE SYSTEM

The input and output relationship of a control system can be expressed by n^{th} order differential equation. If the system is governed by nth order differential equation, then the system is called nth order system.

Also, the order of the system can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in s.

Transfer function T(s) = K $\frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + ...b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_{n-1}s + a_n}$ ...... (3.6)

where, P(s) = Numerator Polynomial

Q(s) = Denominator Polynomial

The order of the system is given by the maximum power of s in the denominator polynomial, Q(s). The order of the system is given by the order of the differential equation governing the system.

Here, Q(s) = $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots a_{n-1}s + a_n$

Now, n is the order of the system

when n=0, the system is zero order system

when n=1, the system is first order system

when n=2, the system is second order system and so on.

3.4 TYPE NUMBER OF THE SYSTEM

The number of poles lying at the origin decides the type number of the system. The type number is specified for loop transfer function G(s)H(s). In general, if N is the number of poles at the origin, then the type number is N.

The loop transfer function can be expressed as a ratio of two polynomials is s.

$G(s)H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)(s+z_2)(s+z_3)......}{s^N(s+p_1)(s+p_2)(s+p_3)......}$ .... (3.7)

where, z_1, z_2, z_3, …. are zeros of transfer function

p_1, p_2, p_3, …. are poles of transfer function

K = Constant

N = Number of Poles at the origin.

The value of N in the denominator polynomial of loop transfer function decides the type number of the system.

If N = 0, then the system is type - 0 system

If N = 1, then the system is type - 1 system and so on.

3.5 RESPONSE OF FIRST ORDER SYSTEM FOR UNIT STEP INPUT

A closed loop system with unity feedback is shown as

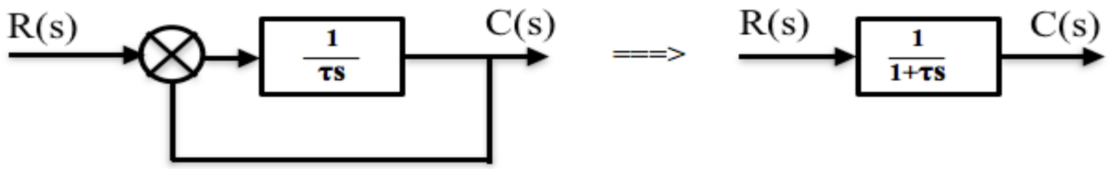


Fig 3.4: Closed loop first order system

First order system may represent RC circuit, thermal system etc. All systems having the same transfer function will exhibit the same output in response to the same input.

The Closed loop transfer function of first order system,

$$\frac{C(s)}{R(s)} = \frac{1}{\tau s+1} \qquad \text{...... (3.8)}$$

Response in s –Domain

$$C(s) = R(s)\left(\frac{1}{\tau s+1}\right) \qquad \text{...... (3.9)}$$

When Unit Step signal is applied

$$R(s) = \frac{1}{s} \qquad \text{...... (3.10)}$$

Substitute equation 3.10 in equation 3.9

$$C(s) = \frac{1}{s(\tau s+1)} \qquad \text{...... (3.11)}$$

Using partial differential expansion method,

$$C(s) = \frac{A}{s} + \frac{B}{\tau s+1} \qquad \text{...... (3.12)}$$

A is obtained by multiplying C(s) by s and letting s=0

$$A = C(s) \times s \,\Big|_{s=0} = \left(\frac{\frac{1}{\tau}}{s\,(s+\frac{1}{\tau})}\right) \times s \,\Big|_{s=0} = \left(\frac{\frac{1}{\tau}}{s+\frac{1}{\tau}}\right) \Big|_{s=0} = \frac{\frac{1}{\tau}}{\frac{1}{\tau}} = 1$$

$$\boxed{A = 1}$$

B is obtained by multiplying C(s) by $\left(s+\frac{1}{\tau}\right)$ and letting $s = -\frac{1}{\tau}$

$$B = C(s) \times s \,\Big|_{s=-1/\tau} = \left(\frac{\frac{1}{\tau}}{s(\,s+\frac{1}{\tau})}\right) \times \left(s+\frac{1}{\tau}\right) \Big|_{s=-1/\tau} = \frac{\frac{1}{\tau}}{s} \,\Big|_{s=-1/\tau} = \frac{\frac{1}{\tau}}{-\frac{1}{\tau}} = -1$$

$$\boxed{B = -1}$$

Substitute A and B in equation 3.12,

$\therefore C(s) = \frac{1}{s} - \frac{1}{\tau s+1}$

On rearranging the denominator, we have

$C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

On taking the Inverse Laplace transform. The response in time domain is given by,

$c(t) = \mathcal{L}^{-1}\, C(s) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}\right\} = 1\text{-}e^{-t/\tau}$

$$c(t) = 1\text{-}e^{-t/\tau} \qquad \text{...... (3.13)}$$

The equation (3.13) is the response of the closed loop first order system for unit step input. For step input of magnitude, A, the equation (3.13) is multiplied by A.

$\therefore$ For Closed loop first order system,
Unit step response c(t) = 1-$e^{-t/\tau}$
Step Response with magnitude A, c(t) = A (1-$e^{-t/\tau}$)

When t=0, $c(0) = 1\text{-}e^{-0/\tau} = 1\text{-}e^{0} = 1\text{-}1 = 0$

When t= τ, $c(\tau) = 1\text{-}e^{-\tau/\tau} = 1\text{-}e^{-1} = 1\text{-}\ 0.36 = 0.632 = 63.2\%$

When t= 2τ, $c(2\tau) = 1\text{-}e^{-2\tau/\tau} = 1\text{-}e^{-2} = 1\text{-}\ 0.135 = 0.864 = 86.4\%$

When t= 3τ, $c(3\tau) = 1\text{-}e^{-3\tau/\tau} = 1\text{-}e^{-3} = 1\text{-}\ 0.049 = 0.950 = 95\%$

When t= 4τ, $c(4\tau) = 1\text{-}e^{-4\tau/\tau} = 1\text{-}e^{-4} = 1\text{-}\ 0.018 = 0.981 = 98.1\%$

When t= 5τ, $c(5\tau) = 1\text{-}e^{-5\tau/\tau} = 1\text{-}e^{-5} = 1\text{-}\ 0.0 = 0.993 = 99\%$

When t= ∞, $c(\infty) = 1\text{-}e^{-\infty/\tau} = 1\text{-}e^{-\infty} = 1\text{-}\ 0 = 1 = 100\%$

Initially the output c(t) is zero and finally it becomes 1. When t = τ, the value of c(t) reaches 63.2% of its total change. In a time of 5τ, the system is assumed to have attained the steady state. Here τ is called Time Constant.

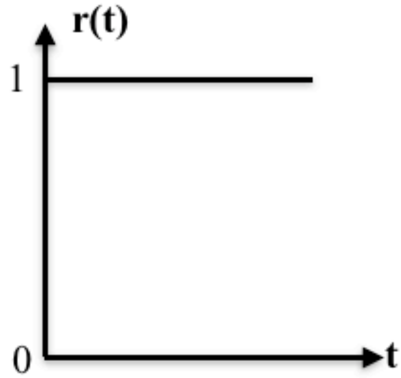


Fig 3.5a : Unit step input

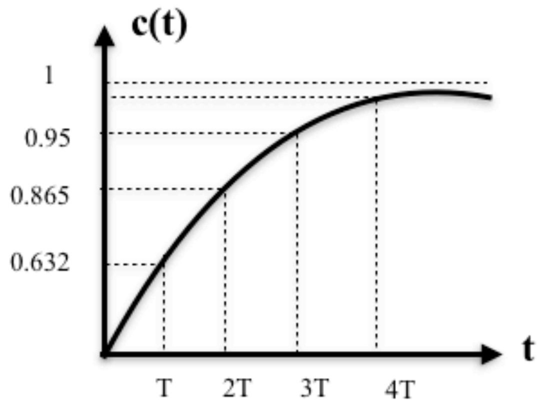


Fig 3.5b : Response for unit step input

Fig 3.5 : Response of first order system to unit step input

3.6 SECOND ORDER SYSTEM

The Closed loop second order system is shown in Fig 3.6

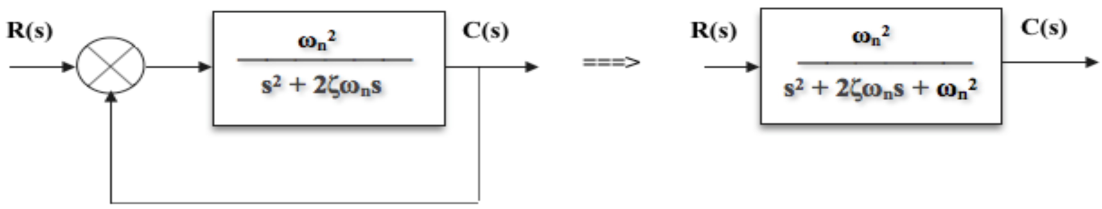


Fig 3.6 : Response of second order system

The Standard form of closed loop transfer function of a second order system is given by,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

The **Damping Ratio** is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends on the value of damping ratio. Depending on the value of the system can be classified into the following four cases,

Case 1: Undamped system, $\zeta = 0$

Case 2: Under damped system, $0 < \zeta < 1$

Case 3: Critically damped system, $\zeta = 1$

Case 4: Over damped system, $\zeta > 1$

The characteristics equation of the second order system is,

$$s^2+2\zeta\omega_n s + \omega_n^2 = 0 \qquad \text{....... (3.14)}$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1, s_2 = \frac{-2\zeta\omega n \pm \sqrt{4\zeta^2\omega_n^2 - 4\,\omega_n^2}}{2} = \frac{-2\zeta\omega n \pm \sqrt{4\omega_n^2(\zeta^2-1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad \text{....... (3.15)}$$

when $\zeta = 0$, $s_1, s_2 = \pm j\omega_n$: $\begin{cases}\text{roots are real and imaginary and}\\ \text{the system is undamped}\end{cases}$ ...... (3.16)

when $\zeta = 1$, $s_1, s_2 = -\omega_n$: $\begin{cases}\text{roots are real and equal and}\\ \text{the system is critically damped}\end{cases}$ ...... (3.17)

when $\zeta > 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$: $\begin{cases}\text{roots are real and unequal and}\\ \text{the system is Over damped}\end{cases}$ ...... (3.18)

when $0 < \zeta < 1$, $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{-1}\,\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$s_1, s_2 = -\zeta\omega_n \pm j\omega_d$: $\begin{cases}\text{roots are Complex conjugate and}\\ \text{the system is Under damped}\end{cases}$ ...... (3.19)

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Here ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

3.6.1 RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad \text{...... (3.20)}$$

For undamped system, $\zeta = 0$, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+\omega_n^2}$ ...... (3.21)

When the input is unit step, $r(t) = 1$ $\quad R(s) = \frac{1}{s}$

$\therefore$ The response in s-domain, $C(s) = R(s)\frac{\omega_n^2}{s^2+\omega_n^2} = \frac{1}{s}\frac{\omega_n^2}{s^2+\omega_n^2}$ ...... (3.22)

By partial Fraction expansion,

$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{B}{s^2+\omega_n^2}$$

A is obtained by multiplying C(s) by s and letting s=0

$A = C(s) \times s\,|_{s=0} = \frac{\omega_n^2}{s(s^2+\omega_n^2)} \times s\,|_{s=0} = \frac{\omega_n^2}{s^2+\omega_n^2}\,|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$ ...... (3.23)

$\boxed{A = 1}$

B is obtained by multiplying C(s) by $(s^2+\omega_n^2)$ and letting $s^2 = -\omega_n^2$ or $s=j\omega_n$.

$B = C(s) \times (s^2+\omega_n^2)\,|_{s=j\omega_n} = \frac{\omega_n^2}{s(s^2+\omega_n^2)} \times (s^2+\omega_n^2)\,|_{s=j\omega_n} = \frac{\omega_n^2}{s}\,|_{s=j\omega_n} = \frac{\omega_n^2}{j\omega_n} = -j\omega_n = -s$

$\boxed{B = -s}$

$\therefore \quad C(s) = \frac{A}{s} + \frac{B}{s^2+\omega_n^2} == \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$

$\boxed{\mathcal{L}\{1\} = \frac{1}{s},\ \mathcal{L}\{\cos\omega t\} = \left(\frac{s}{s^2+\omega_n^2}\right)}$

Time Domain Response, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right\} = 1 - \cos\omega_n t$ ..... (3.24)

The equation (3.24) is the response of undamped closed loop second order system for unit step input. For step input of magnitude, A, the equation (3.24) should be multiplied by A.

$\therefore$ For Closed loop undamped second order system,

Unit Step response $c(t) = 1 - \cos\omega_n t$

Step Response with magnitude, A, $c(t) = A(1 - \cos\omega_n t)$

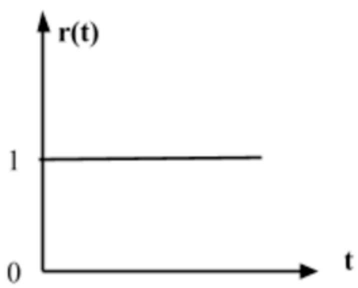


Fig : 3.7a Input

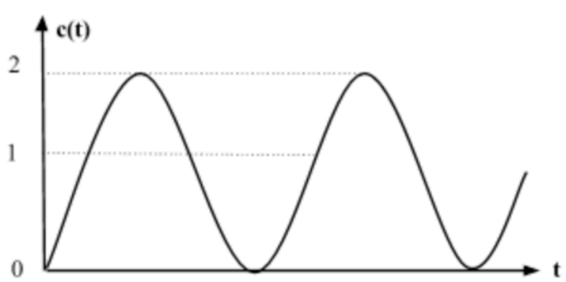


Fig : 3.7b Response

Fig 3.7 : Response of undamped second order system for unit step input

Using equation (3.24), the response of undamped second order system for unit step input is sketched in fig 3.7, and observed that the response is completely oscillatory.

3.6.2 RESPONSE OF UNDERDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system

$$\frac{\mathbf{C(s)}}{\mathbf{R(s)}} = \frac{\boldsymbol{\omega_n}^2}{\mathbf{s^2+2\zeta\omega_n s+\omega_n}^2}$$

For underdamped system, $0 < \zeta < 1$ and roots of the denominator (characteristic equation) are complex conjugate.

The response in s-domain, $C(s) = R(s)\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}$

For unit step input, r(t) = 1 R(s) = $\frac{1}{s}$.

$$C(s) = R(s)\frac{\omega_n^2}{s\,(s^2+2\zeta\omega_n s+\omega_n^2)}$$

By partial Fraction expansion method,

$$C(s) = \frac{\omega_n^2}{s\,(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad \text{...... (3.25)}$$

A is obtained by multiplying C(s) by s and letting s = 0,

$$\therefore A = s \times C(s)|_{s=0} = s\times\frac{\omega_n^2}{s\,(s^2+2\zeta\omega_n s+\omega_n^2)}\Big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

To solve for B and C, cross multiply equation (3.25) and equate like power of s.

On cross multiplication equation (3.25) after substituting A = 1, we get,

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + (Bs + C)s$$

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

Equating the Coefficients of s^2, we get, 0 = 1 + B $\qquad \therefore$ B = -1

Equating the Coefficients of s, we get, $0 = 2\zeta\omega_n + C$ $\qquad \therefore C = -2\zeta\omega_n$

$$\therefore C(s) = \frac{A}{s} + \frac{Bs+C}{(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad \text{...... (3.26)}$$

Let us add and subtract $\zeta^2\omega_n^2$ to the denominator of second term in the equation (3.26)

$$\therefore C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2 - \zeta^2\omega_n^2 + \zeta^2\omega_n^2} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2\,(1-\zeta^2)} = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} \qquad \boxed{\omega_d^2 = \omega_n^2\,(1-\zeta^2)}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} \qquad ......(3.27)$$

Let us multiply and divide by ω_d in the third term of the equation (3.27)

$$\therefore C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}$$

$$= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$\boxed{\omega_d = \omega_n\sqrt{(1-\zeta^2)}}$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega^2}{(s+a)^2+\omega^2}$$

$$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right)$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\omega_d t \times \zeta + \cos\omega_d t\sqrt{1-\zeta^2}\right)$$

Let us express c(t) in a standard form as shown below,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\omega_d t \times \cos\theta + \cos\omega_d t\,\sin\theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta) \qquad ......(3.28)$$

where, $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

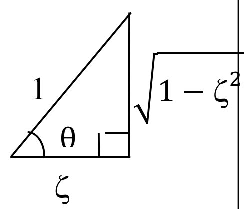
The equation (3.28) is the response of underdamped closed loop second order system for unit step input. For Step input of value, A the equation (3.28) should be multiplied by A.

Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$



$$\therefore \textbf{ For Closed loop underdamped second order system,}$$

$$\text{Unit Step response } c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta); \quad \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\text{Step Response with magnitude,A, } c(t) = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)\right]; \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

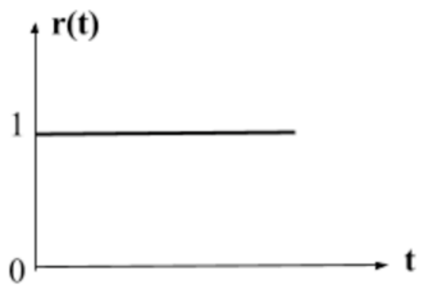


Fig 3.8a : Input

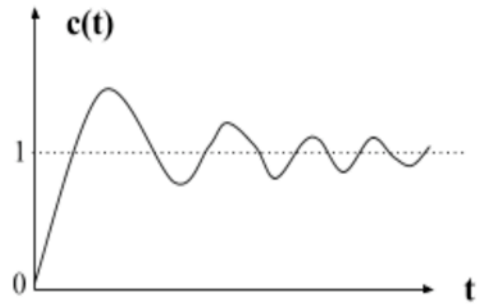


Fig 3.8b : Response

Fig 3.8 Response of underdamped second order system for unit step input

Using equation (3.28) the response of underdamped second order system for unit step input is sketched and observed that the response oscillates before setting to a final value. The oscillations depend on the value of damping ratio.

3.6.3 RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For critical damping ζ=1,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2} \qquad \text{...... (3.29)}$$

When input is unit step, r(t) = 1 R(s) = $\frac{1}{s}$.

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{\omega_n^2}{s(s+\omega_n)^2} \qquad \text{....... (3.30)}$$

By Partial Fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = s \times C(s)|_{s=0} = \frac{\omega_n^2}{s(s+\omega_n)^2}|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = [(s+\omega_n)^2 \times C(s)]\,|_{s=-\omega_n} = (s+\omega_n)^2 \frac{\omega_n^2}{s(s+\omega_n)^2}|_{s=-\omega_n} = \left(\frac{\omega_n^2}{s}\right)|_{s=-\omega_n} = -\omega_n$$

$$C = \frac{d}{ds}[(s+\omega_n)^2 \times C(s)]\,|_{s=-\omega_n} = \frac{d}{ds}\left(\frac{\omega_n^2}{s}\right)|_{s=-\omega_n} = \left(\frac{-\omega_n^2}{s^2}\right)|_{s=-\omega_n} = -1$$

$$\therefore\ C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right\}$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \qquad \text{...... (3.31)}$$

The equation (3.31) is the response of critically damped closed loop second order system for unit step input. For step input of magnitude, A, the equation (3.31) should be multiplied by A.

∴ For closed loop critically damped second order system,

Unit step response, $c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)$

Step response with magnitude A, $c(t) = A\,[1 - e^{-\omega_n t}(1+\omega_n t)]$

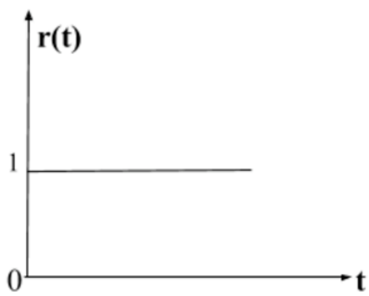


Fig 3.9a : Input

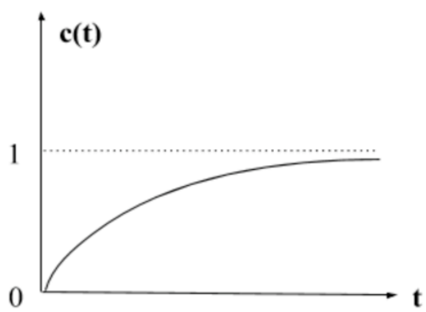


Fig 3.9b : Response

Fig 3.9 Response of critically damped second order system for unit step input

Using equation (3.31), the response of critically damped second order system is sketched as shown in fig 3.9 and observed that the response has have no oscillations.

3.6.4 RESPONSE OF OVER DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For overdamped system $\zeta > 1$. The roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b.

$$s_a, s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} = -\left[\zeta\omega_n \mp \omega_n\sqrt{\zeta^2-1}\right] \quad \text{....... (3.32)}$$

$$\therefore\ s_1 = \left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right] \quad \text{....... (3.33)}$$

$$s_2 = \left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right] \quad \text{...... (3.34)}$$

Let $s_1 = -s_a,\ s_2 = -s_b$

The closed loop transfer function can be written in terms of s_1 and s_2 as shown below,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)} \quad \text{...... (3.35)}$$

When input is unit step is applied, $R(s) = \frac{1}{s}$.

$$\therefore\ C(s) = R(s)\frac{\omega_n^2}{(s+s_1)(s+s_2)} = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

By partial fraction expansion we can write, $C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{(s+s_1)} + \frac{C}{(s+s_2)}$

$$A = s \times C(s)\,|_{s=0} = s \times \frac{\omega_n^2}{s(s+s_1)(s+s_2)}\,|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \zeta^2\omega_n^2+\omega_n^2} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+s_1)\times C(s)\,|_{S=-S1} = (s+s_1)\frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{\omega_n^2}{s(s+s_2)}\,|_{S=-S1} = \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{S_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]S_1} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}\frac{1}{S_1}$$

$$C = (s+s_2)\times C(s)\,|_{S=-S2} = (s+s_2)\frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{\omega_n^2}{s(s+s_1)}\,|_{S=-S2} = \frac{\omega_n^2}{-s_2(-s_2+s_1)} = \frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$= \frac{-\omega_n^2}{S_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n^2}{-\left[2\omega_n\sqrt{\zeta^2-1}\right]S_2} = \frac{\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]S_2} = \frac{\omega_n}{\left[2\sqrt{\zeta^2-1}\right]}\frac{1}{S_2}$$

The response in time domain, c(t) is given by,

$$c(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{S_1}\,\frac{1}{(s+s_1)} + \frac{\omega_n}{\left[2\sqrt{\zeta^2-1}\right]}\,\frac{1}{S_2}\,\frac{1}{(s+s_2)}\right\}$$

$$c(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{S_1}\,e^{-S_1 t} + \frac{\omega_n}{\left[2\sqrt{\zeta^2-1}\right]}\,\frac{1}{S_2}\,e^{-S_2 t}\right\}$$

$$c(t)= 1-\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-S_1 t}}{s_1}-\frac{e^{-S_2 t}}{s_2}\right) \qquad \text{...... (3.36)}$$

The equation (3.36) is the response of overdamped closed loop system for unit step input. For step input of value, A, the equation (3.36) is multiplied by A.

∴ For Closed Loop Overdamped second order system,

Unit step response, $c(t) = 1-\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-S_1 t}}{s_1}-\frac{e^{-S_2 t}}{s_2}\right)$

Step response of magnitude, A, $c(t)= A\left[1-\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-S_1 t}}{s_1}-\frac{e^{-S_2 t}}{s_2}\right)\right]$

where, $s_1 = \left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right] \qquad s_2 = \left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]$

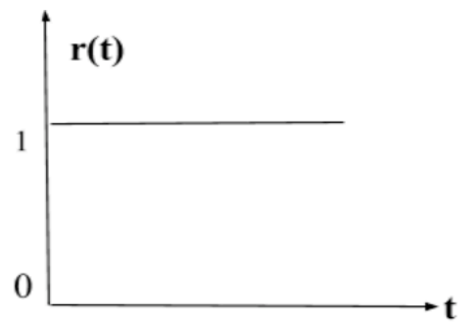


Fig 3.10a : Input

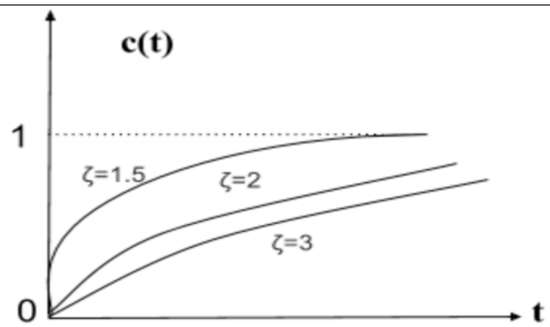


Fig 3.10b : Response

Fig 3.10 Response of over damped second order system for unit step input

Using equation (3.36), the response of overdamped second order system is sketched as shown in fig 3.10 and observed that the response has no oscillations but it takes longer time for the response to reach the final steady value.

EXAMPLE 3.1:

1. Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$, when the input is unit step.

Solution:

Given: $G(s) = \frac{4}{s(s+5)}$, $H(s) = 1$

We know that, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(S)H(s)} = \frac{\frac{4}{s(s+5)}}{1+\frac{4}{s(s+5)}(1)} = \frac{\frac{4}{s(s+5)}}{\frac{s(s+5)+4}{s(s+5)}} = \frac{4}{s^2+5s+4}$

Also, given that the input is unit step $\therefore r(t)=1$ $R(s)= \frac{1}{s}$

$C(s) = R(s)\frac{4}{s^2+5s+4} = \frac{1}{s}\frac{4}{s^2+5s+4} = \frac{4}{s(s^2+5s+4)} = \frac{4}{s(s+1)(s+4)}$

By partial Fraction expansion method, we get

$C(s)= \frac{A}{s}+ \frac{B}{s+1}+ \frac{C}{s+4}$

To Find A, multiply C(s) by s and let s = 0

$A = C(s)\times s|_{s=0} = \frac{4}{(s+1)(s+4)}|_{s=0} = \frac{4}{(1)(4)} = \frac{4}{4} = 1$

To Find B, multiply C(s) by (s+1) and let s = -1

$B= C(s)\times (s+1)|_{s=-1} = \frac{4}{s(s+4)}|_{s=-1} = \frac{4}{(-1(3)} = \frac{4}{-3} = -1.33$

To Find B, multiply C(s) by (s+4) and let s = -4

$C = C(s) \times(s+4)|_{s=-4} = \frac{4}{s(s+1)}|_{s=-4} = \frac{4}{(-4(-3)} = \frac{4}{12} = 0.33$

Substitute A, B and C in equation 3.37,

$C(s)= \frac{1}{s} - \frac{1.33}{s+1} + \frac{0.33}{s+4}$

$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1.33}{s+1} + \frac{0.33}{s+4}\right\}$

RESULT:

Response $c(t) = 1- 1.33e^{-t}+0.33e^{-4t}$

3.7 TIME DOMAIN SPECIFICATION

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses, when they are subjected to inputs or disturbances.

The transient response characteristics of a control system to a unit step input are specified in terms of the following time domain specifications.

1. **Delay time, t_d**
2. **Rise time, t_r**
3. **Peak time, t_p**
4. **Maximum Peak overshoot, M_p**
5. **Settling Time, t_s**

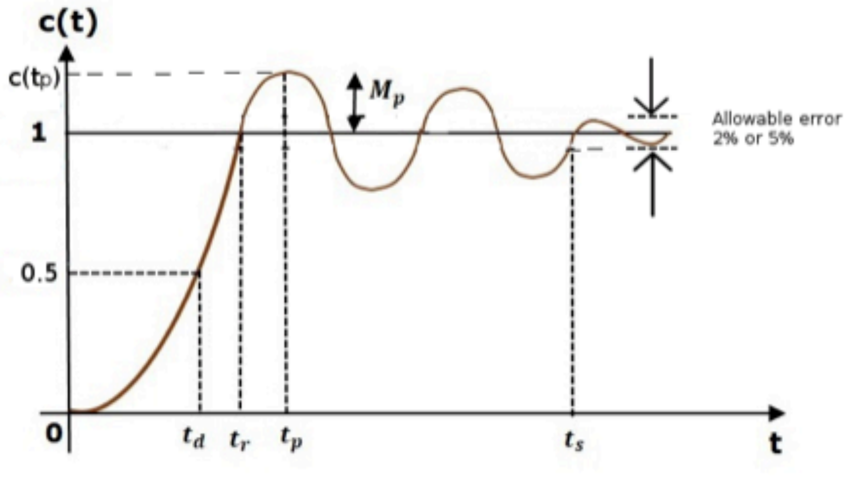


Fig 3.11 Damped Oscillatory response of second order system for unit step input

The time domain specifications are defined as follows

Delay Time (t_d):

It is the time taken for response to reach 50% of the final value for the very first time.

Rise Time (t_r):

It is the time taken for the response to rise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for over damped system, it is the time taken by the response to rise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%. t_r.

$$t_r = \left[\frac{\pi - \theta}{\omega_d}\right] \qquad ..... (3.38)$$

where, $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Peak Time (t_p):

It is the time taken for the response to reach the peak value the very first time. (or) It is the time taken for the response to reach the peak overshoot, M_p.

$$t_p = \left[\frac{\pi}{\omega_d}\right] \qquad ...... (3.39)$$

The damped frequency of oscillation $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Peak Overshoot (M_p):

It is defined as the normalized difference between the time response peak and the steady output.

$$\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 \qquad ...... (3.40)$$

where, c(t) = Peak response at t = t_p,

c (∞) = Final steady state value

$$\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

Settling Time (t_s):

It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2% to 5% of the final value.

$$\mathbf{t_s} = \left[\frac{4}{\zeta\,\omega_n}\right] = 4\tau \text{ (For 2\% error)}$$

$$\mathbf{t_s} = \left[\frac{3}{\zeta\,\omega_n}\right] = 3\tau \text{ (For 5\% error)}$$

In general, for a specified percentage error, settling time can be evaluated using equation.

$$t_s = \left[\frac{-\ln(\%\text{error})}{\zeta\,\omega_n}\right] \qquad ...... (3.41)$$

EXAMPLE 3.2:

The open loop transfer function of a unity feedback system is G(s) = $\frac{20}{s(s+6)}$. Find the following

i. **Time constant**
ii. **Rise time**
iii. **Peak Time**
iv. **Peak overshoot**
v. **Settling Time for 5% tolerance.**

Solution:

Given: $G(s) = \frac{20}{s(s+6)}$, $H(s) = 1$

We know that $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(S)H(s)} = \frac{\frac{20}{s(s+6)}}{1+\frac{20}{s(s+6)}(1)} = \frac{\frac{20}{s(s+6)}}{\frac{s(s+6)+20}{s(s+6)}} = \frac{20}{s^2+6s+20}$

On comparing with the general equation

$$\frac{20}{s^2+6s+20} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Undamped Natural Frequency (ω_n): $\omega_n^2 = 20;\ \omega_n = \sqrt{20} = 4.47$ rad/sec.

Damping Ratio (ζ): $2\zeta\omega_n = 6$; $2\times4.47\times\zeta = 6;\ \zeta = \frac{6}{8.94} = 0.671$

This system is underdamped system as the value of damping ratio is less than 1.

(i) Time Constant, (τ):

$$\tau = \frac{1}{\zeta\omega_n} = \frac{1}{4.47\times0.671} = \frac{1}{2.999} = 0.33 \text{ sec.}$$

Damped Natural Frequency, (ω_d):

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.47\sqrt{1-(0.671)^2} = 4.47\times\sqrt{0.549} = 4.47\times0.741$$

$$\omega_d = 3.314 \text{ rad/sec.}$$

(ii) Rise Time, (t_r):

$$t_r = \left[\frac{\pi-\theta}{\omega_n\sqrt{1-\zeta^2}}\right]$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-(0.671)^2}}{0.671} = \tan^{-1}\frac{0.741}{0.671} = \tan^{-1}1.104 = 47.82°$$

By Converting Degrees to Radians

$\theta = 0.83$ radians

$47.82° = (\frac{\pi}{180} \times 47.82)$ radians= 0.83 radians

$$t_r = \frac{3.14-0.83}{3.314} = \frac{2.31}{3.314} = 0.697 \text{ sec.}$$

(iii) Peak Time, (t_p):

$$t_p = \left[\frac{\pi}{\omega_d}\right] = \left[\frac{3.14}{3.314}\right] = 0.947 \text{ sec.}$$

(iv) Peak Overshoot, (M_p):

$$\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\%M_p = e^{\frac{-(0.671)(3.14)}{0.741}} \times 100 = e^{\frac{-2.106}{0.741}} \times 100 = e^{-2.842} \times 100 = 0.058 \times 100 = 5.8\%$$

(v) Settling Time, (t_s):

$$t_s = -\frac{\ln(\%\text{error})}{\zeta\,\omega_n} = \frac{-\ln(0.05)}{(0.671)(4.47)} = \frac{2.99}{2.9993} = 0.996 \text{ sec.}$$

RESULT:

(i) τ = 0.33 sec (ii) t_r = 0.697 sec (iii) t_p = 0.947 sec

(iv) %M_p = 5.8% (v) t_s = 0.996 sec

3.8 STEADY STATE ERROR

The steady state error is the value of error signal e(t), when t tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearity of system components. The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs.

Consider a closed loop system shown in fig 3.12

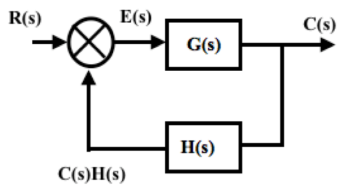


Fig 3.12

Let R(s) = Input signal

E(s) = Error signal

C(s)H(s) = Feedback signal

C(s) = Output signal or response

The error signal, $E(s) = R(s) - C(s)H(s)$ ...... (3.42)

The output signal, $C(s) = E(s)G(s)$ ...... (3.43)

On substituting for C(s) from equation (3.43) in equation (3.42) we get,

$$E(s) = R(s) - [E(s)G(s)]H(s)$$

$$E(s) + E(s)G(s)H(s) = R(s)$$

$$E(s)\,[1 + G(s)H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} \quad \text{...... (3.44)}$$

Let, e(t) = error signal in time domain.

$$\therefore e(t) = L^{-1}\{E(s)\} = \left\{\frac{R(s)}{1+G(s)H(s)}\right\} \quad \text{...... (3.45)}$$

Let e_{ss} = steady state error.

The steady state error is defined as the value of error signal e(t) when t tends to infinity.

$$\mathbf{e_{ss} = \underset{t\to\infty}{Lt}\ e(t)}$$

The final value theorem of Laplace transform states that,

If $F(s) = \mathcal{L}\{f(t)\}$ then, $\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\,F(s)$

Using Final Value theorem,

$$\text{The steady state error } e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s) = \underset{s\to 0}{Lt}\left(\frac{s\,R(s)}{1+G(s)\,H(s)}\right) \quad \text{...... (3.46)}$$

3.9 STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type-0 system will have a constant steady state error when the input is step signal. Type-1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal. For the three cases mentioned above, the steady state error is associated with one of the constants defined as follows.

Positional error constant, $K_p = \underset{s\to 0}{Lt}\ G(s)\,H(s)$ …… (3.47)

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s)$ …… (3.48)

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2\,G(s)H(s)$ …… (3.49)

The K_p, K_v, K_a are in general called static error constants .

3.10 STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

The steady state error $e_{ss} = \underset{s\to 0}{Lt}\left(\frac{s\,R(s)}{1+G(s)\,H(s)}\right)$

When the input is unit step, $R(s)= \frac{1}{s}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\left(\frac{s\,\frac{1}{s}}{1+G(s)\,H(s)}\right) = \underset{s\to 0}{Lt}\left(\frac{1}{1+G(s)\,H(s)}\right) = \frac{1}{1+\underset{s\to 0}{Lt}\,G(s)\,H(s)} = \frac{1}{1+K_p}$$

where, $\boxed{K_p = \underset{s\to 0}{Lt}\,G(s)\,H(s)}$ $\boxed{e_{ss} = \frac{1}{1+K_p}}$ ...... (3.50)

The constant K_p is called the positional error constant.

<u>Type-0 System</u>

$$K_p = \underset{s\to 0}{Lt}\,G(s)\,H(s) = \underset{s\to 0}{Lt}\,K\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = K\frac{z_1.z_2.z_3\ldots}{p_1.p_2.p_3\ldots} = \text{Constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{Constant}$$

Hence in type-0 systems when the input is unit step, there will be a constant steady state error.

<u>Type-1 System</u>

$$K_p = \underset{s\to 0}{Lt}\,G(s)\,H(s) = \underset{s\to 0}{Lt}\,K\,\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{s(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

<u>Type-2 System</u>

$$K_p = \underset{s\to 0}{Lt}\,G(s)\,H(s) = \underset{s\to 0}{Lt}\,K\,\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{s^2(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 2 and above, for unit step input, the value of K_p is infinity and so the steady state error is zero.

3.11 STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

The steady state error $e_{ss} = \underset{s\to 0}{Lt}\left(\frac{s\,R(s)}{1+G(s)\,H(s)}\right)$

When the input is unit ramp, $R(s)= \frac{1}{s^2}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\left(\frac{s\frac{1}{s^2}}{1+G(s)\,H(s)}\right) = \underset{s\to 0}{Lt}\left(\frac{1}{s+s\,G(s)\,H(s)}\right) = \frac{1}{s\,G(s)\,H(s)} = \frac{1}{K_v}$$

where, $\boxed{K_v = \underset{s\to 0}{Lt}\,s\,G(s)\,H(s)}$ $\boxed{\therefore e_{ss} = \frac{1}{K_v}}$ ...... (3.51)

The constant K_v is called velocity error constant.

Type-0 System

$$K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,K\,\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

Hence in type-0 system when the input is unit ramp, the steady state error is infinity.

Type-1 System

$$K_V = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,K\,\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{s(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = K\,\frac{z_1.z_2.z_3\ldots}{p_1.p_2.p_3\ldots} = \text{Constant}$$

$$\therefore e_{ss} = \frac{1}{K_v} = \text{Constant}$$

Hence in type-1 system when the input is unit ramp there will be a constant steady state error.

Type-2 System

$$K_V = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,K\,\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{s^2(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

3.12 STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

The steady state error $e_{ss} = \underset{s\to 0}{Lt}\left(\frac{s\,R(s)}{1+G(s)\,H(s)}\right)$

When the input is unit parabola, $R(s)= \frac{1}{s^3}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\left(\frac{s\frac{1}{s^3}}{1+G(s)\,H(s)}\right) = \underset{s\to 0}{Lt}\left(\frac{1}{s^2+s^2\,G(s)\,H(s)}\right) = \frac{1}{s^2\,G(s)\,H(s)} = \frac{1}{K_a}$$

where, $\boxed{K_a = \underset{s\to 0}{Lt}\ s^2 G(s)\,H(s)}$ $\boxed{\therefore e_{ss} = \frac{1}{K_a}}$ ...... (3.52)

The constant K_a is called Acceleration error constant.

Type-0 System

$$K_a = \underset{s\to 0}{Lt}\ s^2 G(s)\,H(s) = \underset{s\to 0}{Lt}\ s^2\,K\,\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-0 systems for parabolic input, the steady state error is infinity.

Type-1 System

$$K_a = \underset{s\to 0}{Lt}\ s^2 G(s)\,H(s) = \underset{s\to 0}{Lt}\ s^2\,K\,\frac{(s+z_1)(s+z_2)(s+z_3)\ldots\ldots}{s\,(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = 0$$

$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$

Hence in type-1 systems for parabolic input, the steady state error is infinity.

Type-2 System

$K_a = \underset{s\to 0}{Lt}\ s^2 G(s)\ H(s) = \underset{s\to 0}{Lt}\ s^2\ K \frac{(s+z_1)(s+z_2\ )(s+z_3)\ldots\ldots}{s^2(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = K\ \frac{z_1.z_2.z_3\ldots}{p_1.p_2.p_3\ldots} = \text{Constant}$

$\therefore e_{ss} = \frac{1}{K_a} = \text{Constant.}$

Hence in type-2 system when the input is unit parabolic there will be a constant steady state error.

Type-3 System

$K_a = \underset{s\to 0}{Lt}\ s^2 G(s)\ H(s) = \underset{s\to 0}{Lt}\ s^2 K\ \frac{(s+z_1)(s+z_2\ )(s+z_3)\ldots\ldots}{s^3(s+p_1)(s+p_2)(s+p_3)\ldots\ldots} = \infty$

$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$

In systems with type number 3 and above, for unit parabolic input, the value of K_a is infinity and so the steady state error is zero.

Table: 3.1 Static error constant for various type number of systems

Error Constant	Type 0	Type 1	Type 2	Type 3
K_p	Constant	∞	∞	∞
K_v	0	Constant	∞	∞
K_a	0	0	Constant	∞

Table: 3.2 Steady state error for various types of input

Input signal	Type 0	Type 1	Type 2	Type 3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

EXAMPLE 3.2:

Determine Positional, velocity and acceleration error constants for the following unity feedback system for which the open loop transfer function is/are

1. $G(s) = \frac{20}{(0.5s+1)(s+10)}$ 2. $G(s) = \frac{K}{s(s^2+2\zeta\omega_n+\omega_n^2)}$ 3. $G(s) = \frac{K(s+2)}{s(s^3+7s^2+12s)}$

Solution:

1. $G(s) = \frac{20}{(0.5s+1)\,(s+10)}$

For unity feedback system, H(s) = 1

∴ Loop Transfer Function G(s)H(s) = G(s)

Positional error constant

$K_p = \underset{s\to0}{Lt}\, G(s)\, H(s) = \underset{s\to0}{Lt} \frac{20}{(0.5s+1)\,(s+10)} = 20/10 = 2$

$K_p = 2$

Velocity error constant

$K_v = \underset{s\to0}{Lt}\, sG(s)\, H(s) = \underset{s\to0}{Lt}\, s \frac{20}{(0.5s+1)\,(s+10)} = 0$

$K_v = 0$

Acceleration error constant

$K_a = \underset{s\to0}{Lt}\, s^2 G(s)\, H(s) = \underset{s\to0}{Lt}\, s^2 \frac{20}{(0.5s+1)\,(s+10)} = 0$

$K_a = 0$

2. $G(s) = \frac{K}{s(s^2+2\zeta\omega_n+\omega_n^2)}$

For unity feedback system, H(s) = 1

∴ Loop Transfer Function G(s)H(s) = G(s)

Positional error constant

$K_p = \underset{s\to0}{Lt}\, G(s)\, H(s) = \underset{s\to0}{Lt} \frac{K}{s(s^2+2\zeta\omega_n+\omega_n^2)} = K/0 = \infty$

$K_p = \infty$

Velocity error constant

$K_v = \underset{s\to0}{Lt}\, s\, G(s)\, H(s) = \underset{s\to0}{Lt}\, s \frac{K}{s(s^2+2\zeta\omega_n+\omega_n^2)} = \frac{K}{\omega_n^2}$

$K_v = \frac{K}{\omega_n^2}$

Acceleration error constant

$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s\to 0}{Lt}\, s^2 \frac{K}{s(s^2+2\zeta\omega_n+\omega_n^2)} = s\frac{K}{(s^2+2\zeta\omega_n+\omega_n^2)} = 0$

$K_a = 0$

3. $G(s) = \frac{K(s+2)}{s(s^3+7s^2+12s)}$

For unity feedback system, H(s) = 1

∴ Loop Transfer Function G(s)H(s) = G(s)

Positional error constant

$K_p = \underset{s\to 0}{Lt}\, G(s)\, H(s) = \underset{s\to 0}{Lt} \frac{K(s+2)}{s(s^3+7s^2+12s)} = 2K/0 = \infty$

$K_p = \infty$

Velocity error constant

$K_v = \underset{s\to 0}{Lt}\, s\, G(s)\, H(s) = \underset{s\to 0}{Lt}\, s \frac{K(s+2)}{s(s^3+7s^2+12s)} = 2K/0 = \infty$

$K_v = \infty$

Acceleration error constant

$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s\to 0}{Lt}\, s^2 \frac{K(s+2)}{s(s^3+7s^2+12s)} = \underset{s\to 0}{Lt}\, s^2 \frac{K(s+2)}{s^2(s^2+7s+12)} = \underset{s\to 0}{Lt} \frac{K(s+2)}{(s^2+7s+12)}$

$K_a = 2K/12$

$K_a = K/6$

EXAMPLE 3.3:

For a unity feedback control system, the open loop transfer function, $G(s) = \frac{10(s+2)}{s^2(s+1)}$.

Find (i) The position, velocity and acceleration error constants.

(ii) The steady state error when the input is R(s), where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

i. **To Find Static error constants**

For a unity feedback system, H(s) = 1

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)\, H(s) = \underset{s\to 0}{Lt}\, G(s) = \underset{s\to 0}{Lt} \frac{10(s+2)}{s^2(s+1)} = \infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, s\, G(s)\, H(s) = \underset{s\to 0}{Lt}\, s\frac{10(s+2)}{s^2(s+1)} = \underset{s\to 0}{Lt}\, s\frac{10(s+2)}{s(s+1)} = \infty$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s\to 0}{Lt}\, s^2 \frac{10(s+2)}{s^2(s+1)} = \underset{s\to 0}{Lt} \frac{10(s+2)}{(s+1)} = 20$

ii. To Find Steady state error

The error signal in s-domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Given that, $R(s)= \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

$$\therefore E(s) = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^2}}{1+\frac{10(s+2)}{s^2(s+1)}} = = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{\frac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+10(s+2)}\right\} = 0 - 0 + \frac{1}{60}$$

$$e_{ss} = \frac{1}{60}$$

RESULT:

(i) **Position error constant, $K_p = \infty$**

Velocity error constant, $K_v = \infty$

Acceleration error constant, $K_a = 20$

(ii) **When, $R(s)= \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$, Steady state error, $e_{ss} = \frac{1}{60}$**

REVIEW QUESTIONS

PART A

1. What is the importance of test signals?
2. What is transient and steady state response?
3. Name the test signals in control system.
4. Define: Damping ratio.
5. How the system is classified according to the value of damping?
6. Sketch the step response of a 1st order system.
7. Sketch the response of a second order undamped system.

8. Write the expression of a damped frequency of oscillations.

9. List the time domain specifications.

10. Define Delay time.

11. Define Rise time.

12. Define time.

13. Define Settling time.

14. Define Peak overshoot

15. What is steady state error.

16. Define: Positional error constant.

17. Define: Velocity error constant.

18. Define: Acceleration error constant.

19. Write the Final Value Theorem

20. Write the Time Domain Response of First order system to unit step input.

PART-B

21.Define Step signal and give its graphical representation and mathematical expression.

22. Define: Ramp signal and give its graphical representation and mathematical expression.

23. Define: Parabolic signal and give its graphical representation and mathematical expression.

24. What is time response? Write the expression for time response in ‘s’ domain and time domain.

25. The transfer function of a system is 10/1+s. Find the steady state error to step input When operated as unity feedback.

26. For unit step input, a system with forward transfer function $G(s)=20/s^2$ and feedback path transfer function is $H(s)=(s+5)$. Find the steady state output.

PART- C

27. For a unity feedback system, G(s)=36/s(s+0.72). Determine the characteristic equation and hence calculate damping ratio, peak time, settling time, peak overshoot and number of cycles completed before output settles for unit step input.

28. The open loop transfer function of a unity feedback system is given by G(s)=40/s(0.2s+1). Determine the steady state error using constants.

29. The response of control system has an overshoot of 30% for a step input and overshoot takes place 0.05 seconds later after the excitation is applied. Find the second order transfer function to achieve this.

30. The open loop transfer function of a system is $G(s)=20 /(s+1)(0.2s+1)$ and feedback path transfer function is $H(s) = 1/5$. Determine the natural frequency of oscillations, damped frequency of oscillations, damping ratio, maximum overshoot and settling time for 2% tolerance band.

31. The following expression denotes the time response of a servomechanism

$c(t) = 1 + 0.2e^{-60t} – 1.2\ e^{-10t}$. Obtain the expression for the closed loop transfer function of the system. Determine the undamped frequency and damping ratio.

32. Consider a unity feedback system with a closed loop transfer function

$C(s)/R(s)= (Ks +b)/(s^2+as+b)$. Determine the open loop transfer function $G(s)$. Show that the steady state error with unit ramp is given by $(a-K) / b$

UNIT IV

FREQUENCY RESPONSE

4. 1 SINUSOIDAL TRANSFER FUNCTION AND FREQUENCY RESPONSE

The response of a system for the sinusoidal input is called sinusoidal response. The ratio of sinusoidal response and sinusoidal input is called sinusoidal transfer function of the system and in general, it is denoted by T (jω). The sinusoidal transfer function is the frequency domain representation of the system, and so it is also called frequency domain transfer function. The sinusoidal transfer function can be obtained as shown below.

1. Construct a physical model of a system using basic elements/parameters.
2. Determine the differential equations governing the system from the physical model of the system.
3. Take Laplace transform of differential equations to convert them to s-domain equation.
4. Determine s-domain transfer function, T(s), which is ratio of s-domain output and input.
5. Determine the frequency domain transfer- function, T (jω) by replacing s by jω in the s-domain transfer function, T(s).

Frequency Response:

The frequency domain transfer function T (jω) is a complex function of ω. Hence it can be separated into magnitude function and phase function. Now, the magnitude and phase functions are real functions and they are called frequency response.

The frequency response can be evaluated for open loop system and closed loop system. The frequency domain transfer function of open loop and closed loop systems can be obtained from the s-domain transfer function by replacing s by jω shown below.

Open loop transfer function G(s) $\xrightarrow{s=j\omega}$ $G(j\omega) = |G(j\omega)| \angle G(j\omega)$

Closed loop transfer function $\frac{C(j\omega)}{R(j\omega)} = M(j\omega) = |M(j\omega)| \angle M(j\omega)$

The advantages of frequency response analysis are the following.

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of their open loop frequency response.
2. The practical testing of systems can be easily carried out with available sinusoidal signal generators and precise measurements
3. The transfer function of complicated systems can be determined experimentally by frequency response tests.

4. The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.
5. When the system is designed by use of the frequency response analysis, the effects of noise disturbance and parameter variations are relatively easy to visualize and incorporate corrective measures.
6. The frequency response analysis and designs can be extended to certain nonlinear control systems.

4.2 FREQUENCY DOMAIN SPECIFICATIONS

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

The frequency domain specifications are,

1. Resonant peak, M_r
2. Resonant Frequency, ω_r
3. Bandwidth
4. Cut-off rate
5. Gain margin, K_g
6. Phase margin, $\boldsymbol{\gamma}$

Resonant Peak:

The maximum value of the magnitude of closed loop transfer function is called the resonant peak. A large resonant peak corresponds to a large overshoot in transient response.

Resonant Frequency:

The frequency at which the resonant peak occurs is called resonant frequency. This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of transient response.

Bandwidth:

The Bandwidth is the range of frequencies for which normalized gain of the system is more than -3db. The frequency at which the gain is -3db is called cut-off frequency. Band width is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut-off frequency. The Bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time. A large-bandwidth corresponds to a small rise time or fast response.

Cut-off Rate:

The slope of the log-magnitude curve near the cut off frequency is called cut-off rate. The cut -off rate indicates the ability of the system to distinguish the signal from noise.

Gain Margin, K_g:

The gain margin, K_g is defined as the reciprocal of the magnitude of open loop transfer function at phase cross over frequency.

$$\textbf{Gain Margin, } K_g = \frac{1}{|G(j\omega)|_{(\omega_{pc})}}$$

The gain margin in db can be expressed as,

$$K_g \textbf{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega)|_{(\omega_{pc})}}$$

Phase cross-over frequency, ω_{pc}.:

The frequency at which the phase of open loop transfer function is 180° is called phase cross-over frequency.

Phase margin, γ:

The phase margin is defined as the additional phase lag to be added at the gain cross over frequency to bring the system to the verge of instability.

$$\textbf{Phase margin } \gamma = 180° + \varphi_{gc}$$

Gain cross over frequency ω_{gc}:

The frequency at which the magnitude of the open loop transfer function is unity (or it is the frequency at which the db magnitude is zero) is called gain cross over frequency ω_{gc}.

The phase margin is obtained by adding 180° to the phase angle φ of the open loop transfer function at the gain cross over frequency.

4.3 BODE PLOT

The Bode plot is a frequency response plot of the sinusoidal transfer function of a system.

A Bode plot consists of two graphs.

1. **One is a plot of the magnitude of a sinusoidal transfer function versus logω,**
2. **The other is a plot of the phase angle of a sinusoidal transfer function versus logω.**

The Bode plot can be drawn for both open loop and closed loop system. Usually the bode plot is drawn for open loop system. The standard representation of the logarithmic magnitude of open loop transfer function of G(jω) is 20 log$|G(j\omega)|$.

Where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel, usually abbreviated in db. The curves are drawn on semi log paper using the log scale (abscissa) for frequency and the linear scale (ordinate) for either magnitude (in decibels) or phase angle (in degrees).

The main advantage of the bode plot is that multiplication of magnitudes can be converted into addition. Also, a simple method for sketching an approximate log-magnitude curve is available.

Consider the open loop transfer function, $G(s) = \dfrac{K(1+sT_1)}{s(1+sT_2)(1+sT_3)}$

$$G(j\omega) = \frac{K(1+j\omega T_1)}{s(1+j\omega T_2)(1+j\omega T_3)}$$

4.3.1 PROCEDURE FOR MAGNITUDE PLOT OF BODE PLOT

From the analysis of previous sections, the following conclusions can be obtained.

1. The constant gain K, integral and derivative factors contribute gain (magnitude) at all frequencies.

2. In approximate plot the first, quadratic and higher order factors contribute gain (magnitude) only when the frequency is greater than the comer frequency. Hence the low frequency response up to the lowest comer frequency is decided by K or $K / (j\omega)^n$ or $K(j\omega)^n$ term. Then at every comer frequency, the slope of the magnitude plot is altered by the first, quadratic and higher order terms. Therefore the magnitude plot can be started-with K or $K/(j\omega)^n$ or $K(j\omega)^n$ term and, then the db magnitude of every first and higher order terms are added one by one in the increasing order of the corner frequency.

This is illustrated in the following example.

$$G(s) = \frac{K(1+ST1)^2}{S(1+ST2)(1+ST3)}$$

$$G(j\omega) = \frac{K(1+(j\omega T1)^2}{S(1+(j\omega T2)(1+(j\omega T3)}$$

Let, $T_1<T_3<T_2$

The corner frequencies are $\omega c1=1/T_1$; $\omega c2=1/T_2$; $\omega c3=1/T_3$.

Let, $\omega c_1<\omega c_3<\omega c_2$, The magnitude plot of the individual terms of G(jω), and their combined magnitude plot are shown in figure.

The step by step procedure for plotting the magl1itude plot is given below:

Step1: Convert the transfer function into Bode form or time constant form. The Bode form of the transfer function is,

$$G(s) = \frac{K(1+sT1)}{s(1+sT2)(1+\frac{s^2}{(\omega_n)^2}+2\delta\frac{s}{\omega_n})}$$

$$G(j\omega) = \frac{K(1+j\omega T_2)}{j\omega(1+j\omega T_2)(1-\frac{\omega^2}{\omega_n{}^2}+j2\delta\frac{\omega}{\omega_n})}$$

Step 2: List the corner frequencies in the increasing order and prepare a table as shown below.

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec

In the above table, enter K or K / $(j\omega)^n$ or $K(j\omega)^n$ as the first term and the other terms in the increasing order of comer frequencies. Then enter the comer frequency, slope contributed by each term and change in slope at every corner frequency.

Step 3: Choose an arbitrary frequency (1) which is lesser than the lowest corner frequency. Calculate the db magnitude of K or K / $(j\omega)^n$ or $K(j\omega)^n$ at ω_l and at the lowest comer frequency.

Step 4: Then calculate the gain (db magnitude) at every corner frequency one by one by using the formula,

Gain at ω_y = change in gain from ω_x to ω_y + Gain at ω_x

= [slope from ω_x to ω_y+log(ω_y/ ω_x)] + gain at ω_x

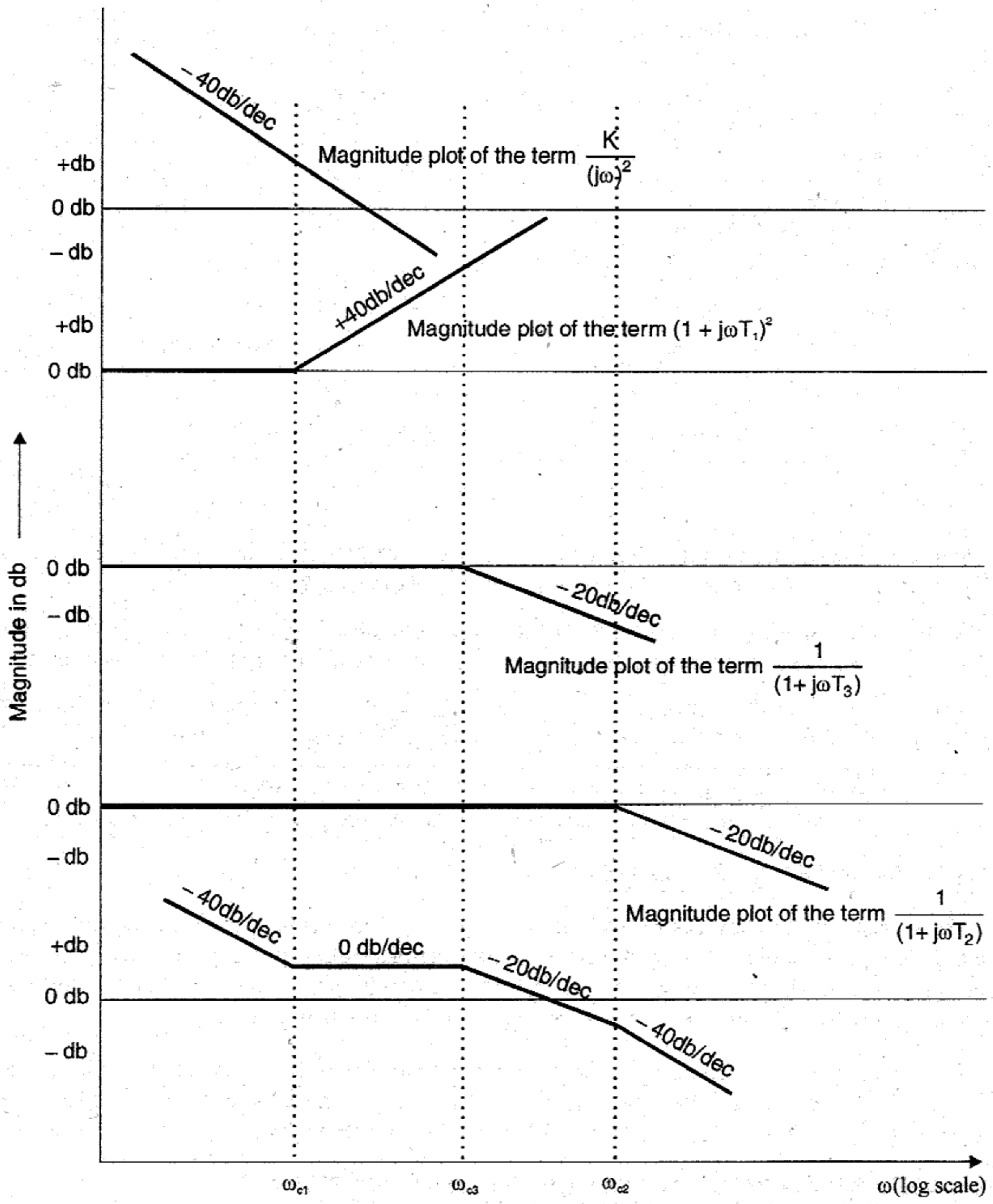


Fig 4.1

Step 5: Choose an arbitrary frequency, ω_h which is greater than the highest frequency. Calculate the gain at ω_h by using the formula in step 4.

Step 6: In a semi log graph sheet mark the required range of frequency on x-axis and the range of db on y-axis (ordinary scale) after choosing proper unit.

Step 7: Mark all the points obtained in steps 3, 4, and 5 on the graph and join the plot by straight lines. Mark the slope at every part of the graph.

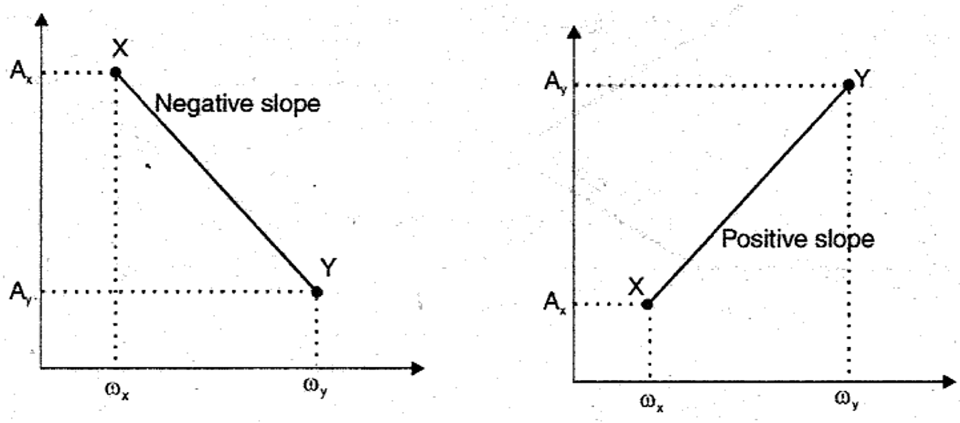


Fig 4.2

4.3.2 PROCEDURE FOR CONSTRUCTING PHASE PLOT OF BODE PLOT:

The phase plot is an exact plot and no approximations are made while drawing the phase plot. Hence the exact phase angles of G(jω) are computed for various values of ω and tabulated. The choice of frequencies is preferably chosen for the magnitude plot. Usually the magnitude plot and phase plot are drawn in a single semi log sheet on a common frequency scale. Take another y axis in the graph where the magnitude plot is drawn and in this y-axis, mark the desired range of phase angles after choosing proper units from the tabulated values of ω and phase angles, mark all the points on the graph. Join the points by a smooth curve.

4.3.3 DETERMINATION OF GAIN MARGIN AND PHASE MARGIN OF BODE PLOT:

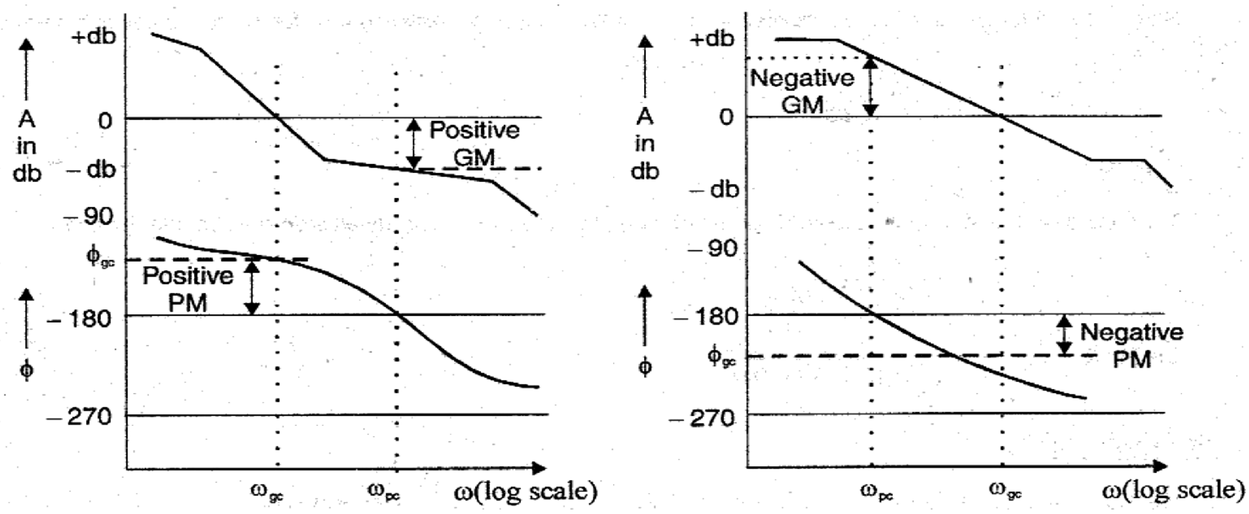


Fig 4.3

The gain margin in db is given by the negative of db magnitude of G(jω) at the phase cross over frequency, ω_{pc}. The ω_{pc} is the frequency at which the phase of G(jω) is 180°. If the db magnitude of G(jω) is negative then the gain margin is positive and vice versa.

Let ϕ_{gc} be the phase angle of G(jω) at gain cross over frequency ω_{gc}. The ω_{gc} is the frequency at which the magnitude of G(jω) is zero. Now the phase margin is given by γ=180°+ ϕ_{gc}. If ϕ_{gc} is less than -180° then the phase margin is positive and vice versa.

The positive and negative gain and phase margins are illustrated in above figure 4.3.

4.3.4 GAIN ADJUSTMENT IN BODE PLOT:

In the open loop transfer function G(jω) the constant K contributes only magnitude. Hence by changing the value of K the system-gain can be adjusted to meet the desired specifications. The desired specifications are gain margin, phase margin. In a system transfer function if the value of K required to be estimated to a desired specification then draw the-bode plot of the system with K = 1. The constant K can add, 20 log K to every point of the magnitude plot and due to this addition, the magnitude plot will shift vertically up or down. Hence shift the magnitude plot vertically up or down to meet the desired specification. Equate the vertical distance by which the magnitude plot is shifted to 20logK and solve for K.

Let, x = change in db

Now, 20logK = x;

Log K = x/20;

K = $10^{x/20}$

EXAMPLE 4.1

Sketch Bode plot for the following transfer function and determine gain cross over frequency and phase cross over frequency.

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$

SOLUTION

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in the given transfer function.

$$G(j\omega) = \frac{10}{j\omega(1+0.4j\omega)(1+0.1j\omega)}$$

MAGNITUDE PLOT:

The corner frequencies are, $\omega c_1 = 1/0.4 = 2.5$ rad/sec

$\omega c_2 = 1/0.1 = 10$ rad/sec

The various terms of $G(j\omega)$ are listed in Table-1 in the increasing order of their corner frequency. Also, the table shows the slope contributed by each term and the change in slope at the comer frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{10}{j\omega}$	–	– 20	
$\frac{1}{1+j0.4\omega}$	$\omega_{c1}=\frac{1}{0.4}=2.5$	– 20	$-20-20=-40$
$\frac{1}{1+j0.1\omega}$	$\omega_{c2}=\frac{1}{0.1}=10$	– 20	$-40-20=-60$

Chose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and chose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let ω_l=0.1 rad/sec and ω_h=50 rad/sec.

Let A= $|G(j\omega)|$ in db.

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

At $\omega=\omega_l$, A= $20\log\left|\frac{10}{j\omega}\right|$ = 20 log (10/0.1) = 40db.

At $\omega=\omega_{c1}$, A= $20\log\left|\frac{10}{j\omega}\right|$ = 20 log (10/2.5 = 12db.

At $\omega=\omega_{c2}$, A= [slope from ω_{c1} to $\omega_{c2}\times\log\frac{\omega_{c1}}{\omega_{c2}}$] + A (at $\omega=\omega_{c1}$)

= -40×log (10/2.5) +12 = -12db.

At $\omega=\omega_h$, A= [slope from ω_{c2} to $\omega_h\times\log\frac{\omega_h}{\omega_{c2}}$] + A (at $\omega=\omega_{c2}$)

= -60×log (50/10) +(-12) = -54db.

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot in a semi log graph sheet choose a scale of 1unit-10 db. on y-axis. The frequencies are marked in decades from 0.1 to 100 r/s on logarithmic scales in x-axis, Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT:

The phase angle of $G(j\omega)$ as a function of ω is given by,

$$\phi = -90° - \tan^{-1} 0.4\omega - \tan^{-1} 0.1\omega$$

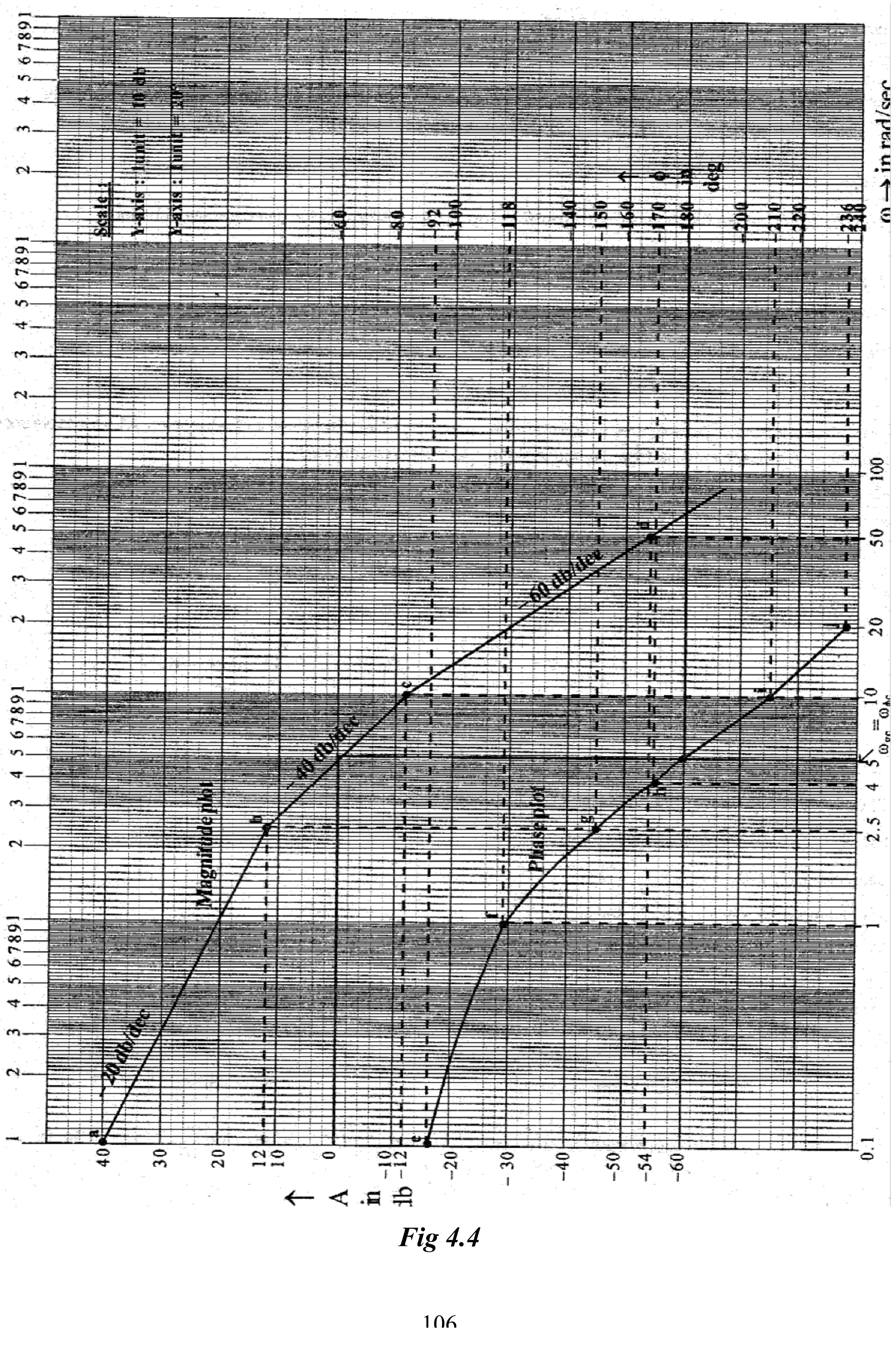


Fig 4.4

The phase angles of various values of ω are calculated and listed in table 2.

TABLE:2

ω rad/sec	$\tan^{-1} 0.4\,\omega$ deg	$\tan^{-1} 0.1\,\omega$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.1	2.29	0.57	$-92.86 \approx -92$	e
1	21.80	5.71	$-117.5 \approx -118$	f
2.5	45.0	14.0	$-149 \approx -150$	g
4	57.99	21.8	$-169.79 \approx -170$	h
10	75.96	45.0	$-210.96 \approx -210$	i
20	82.87	63.43	$-236.3 \approx 236$	j

On the same semi log graph sheet, choose a scale of 1 unit=20° on the y-axis on the right side of semilog graph sheet. Mark the phase angles on the graph sheet and join the graph by smooth curve.

RESULT:

1. Gain cross over frequency = 5 rad/sec
2. Phase cross over frequency = 5 rad/sec.

EXAMPLE 4.2

Sketch Bode plot for the following transfer function and determine system gain K for the gain cross over frequency to be 5r/s. $G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$

SOLUTION

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in the given transfer function.

Let, K=1, we get, $G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$

MAGNITUDE PLOT:

The corner frequencies are,

$\omega c_1 = 1/0.2 = 5$ rad/sec

$\omega c_2 = 1/0.02 = 50$ rad/sec

The various terms of G(jω) are listed in Tatile-1 in the increasing order of their corner frequencies. Also, the table shows the slope contributed by each term and the change in slope at the comer frequency.

Chose a low frequency ω_l such that $\omega_{l} < \omega_{c1}$ and chose a high frequency ω_h such that $\omega_{h} > \omega_{c2}$.

Let ω_l=0.5 rad/sec and ω_h =100 rad/sec.

Let A= $|G(j\omega)|$ in db.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$(j\omega)^2$	–	+40	
$\frac{1}{1+j0.2\omega}$	$\omega_{c1}=\frac{1}{0.2}=5$	–20	$40-20=20$
$\frac{1}{1+j0.02\omega}$	$\omega_{c2}=\frac{1}{0.02}=50$	20	$20-20=0$

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

At $\omega=\omega_l$, A= $20\log|(j\omega)^2|$ = 20 log $(0.5)^2$ = -12db.

At $\omega=\omega_{c1}$, A= $20\log|(j\omega)^2|$ = 20 log $(5)^2$ = 28db.

At $\omega=\omega_{c2}$, A= [slope from ω_{c1} to $\omega_{c2}\times\log\frac{\omega c2}{\omega c1}$] + A (at $\omega=\omega_{c1}$)

= 20×log (50/5) + 28 = 48db.

At $\omega=\omega_h$, A= [slope from ω_{c2} to $\omega_h\times\log\frac{\omega h}{\omega c2}$] + A (at $\omega=\omega_{c2}$)

= 0×log (100/50) + (48) = 48db.

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot in a semi log graph sheet choose a scale of 1unit-10 db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis, Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT:

The phase angles of various values of ω are calculated and listed in table 2.

The phase angle of $G(j\omega)$ as a function of ω is given by,

$$\boldsymbol{\phi} = -180°-\tan^{-1} 0.2\omega-\tan^{-1} 0.02\omega$$

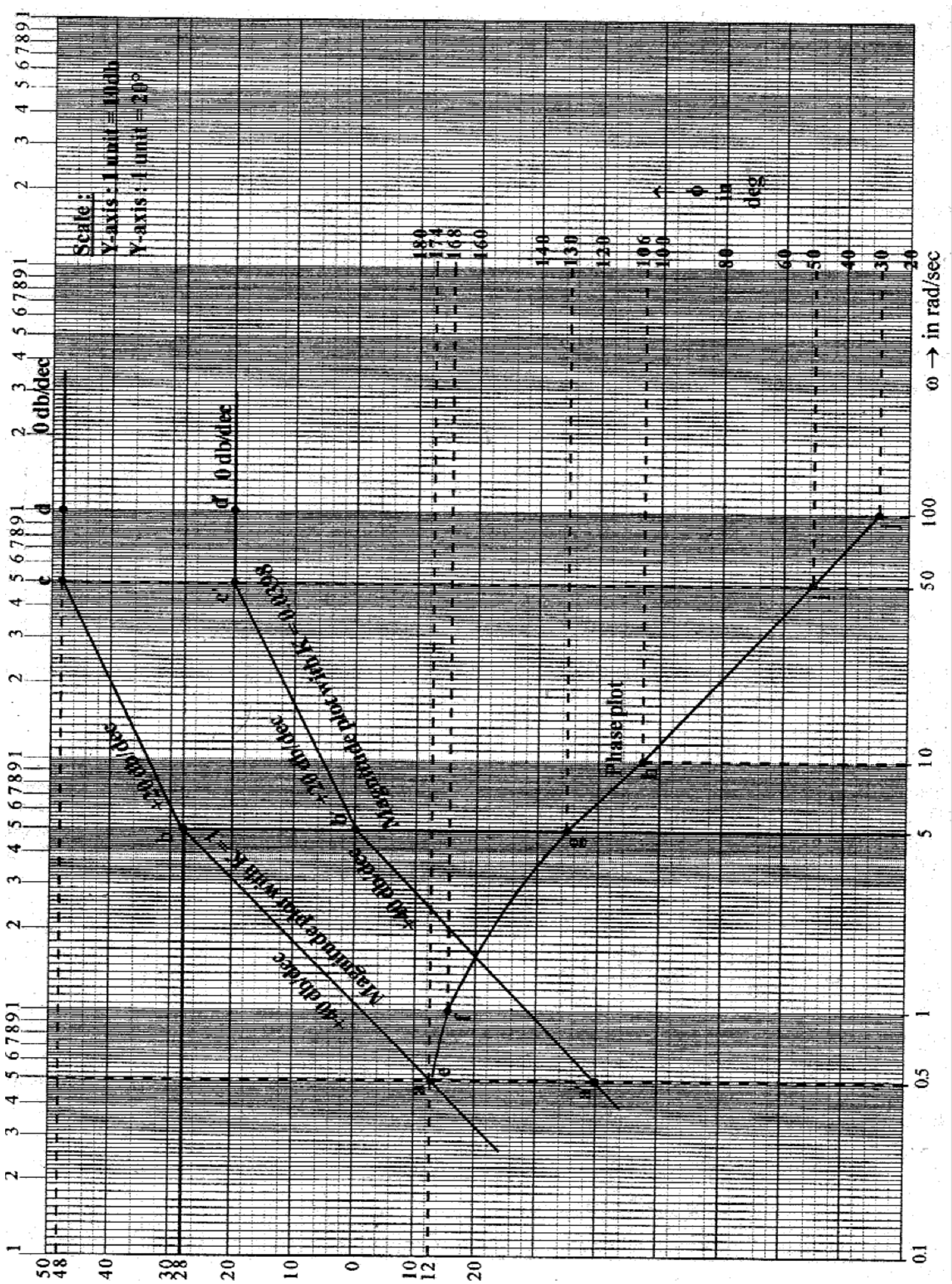

Scale :
Y-axis : 1 unit = 10db
Y-axis : 1 unit = 20°
0 db/dec
Magnitude plot with K = 1
Phase plot
ω → in rad/sec
φ in deg

Fig 4.5

TABLE-2

ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} 0.02\omega$ deg	$\phi = \angle G(j\omega)$ deg	Point in phase plot
0.5	5.7	0.6	$173.7 \approx 174$	e
1	11.3	1.1	$167.6 \approx 168$	f
5	45	5.7	$129.3 \approx 130$	g
10	63.4	11.3	$105.3 \approx 106$	h
50	84.3	45	$50.7 \approx 50$	i
100	87.1	63.4	$29.5 \approx 30$	j

On the same semi log graph sheet choose a scale of 1 unit=20° on the y-axis on the right side of semi log graph sheet. Mark the phase angles on the graph sheet and join the graph by smooth curve.

CALCULATION OF GAIN K:

The gain cross over frequency is = 5 rad/sec. At ω= 5 rad/sec the gain is 28db.

At the gain cross over frequency the gain should be 0. Hence to every point of magnitude the gain -28db should be added.

This addition of gain will shift the plot downward. The magnitude correction is independent of frequencies.

Hence the gain is calculated by,

$$20 \log K = -28$$

$$\text{Log } K = \frac{-28}{20}$$

$$K = 10^{\frac{-28}{20}}$$

$$K = 0.0398$$

EXAMPLE 4.3

Sketch Bode plot for the following transfer function $G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$

SOLUTION

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in the given transfer function.

Let, K=1 get, $G(j\omega) = \frac{5(1+2j\omega)}{(1+4j\omega)(1+0.25j\omega)}$

MAGNITUDE PLOT:

The corner frequencies are,

$\omega c_1 = 1/4 = 0.25$ r/s

$\omega c_2 = 1/2 = 0.5$ r/s

$\omega c_3 = 1/0.25 = 4$ r/s

The various terms of $G(j\omega)$ are listed in Table-1 in the increasing order of their corner frequency. Also, the table shows the slope contributed by each term and the change in slope at the comer frequency.

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and chose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let ω_l=0.1 r/s and ω_h =10 r/s.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/deg
5	–	0	–
$\frac{1}{1+j4\omega}$	$\omega_{c1} = \frac{1}{4} = 0.25$	–20	$0-20=-20$
$1+j2\omega$	$\omega_{c2} = \frac{1}{2} = 0.5$	20	$-20+20=0$
$\frac{1}{1+j0.25\omega}$	$\omega_{c3} = \frac{1}{0.25} = 4$	–20	$0-20=-20$

Let A= $|G(j\omega)|$ in db.

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

At $\omega = \omega_l$, $A = 20\log|(j\omega)| = 20\log(5) = 14$db.

At $\omega = \omega_{c1}$, $A = 20\log|(j\omega)| = 20\log(5) = 14$db.

At $\omega = \omega_{c2}$, $A = [\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega c2}{\omega c1}] + A\ (\text{at } \omega = \omega_{c1})$

$= -20\times\log(0.5/0.25) + 14 = 8$db.

At $\omega = \omega_h$, $A = [\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega h}{\omega c2}] + A\ (\text{at } \omega = \omega_{c2})$

$= 0\times\log(4/0.5) + (8) = 8$db.

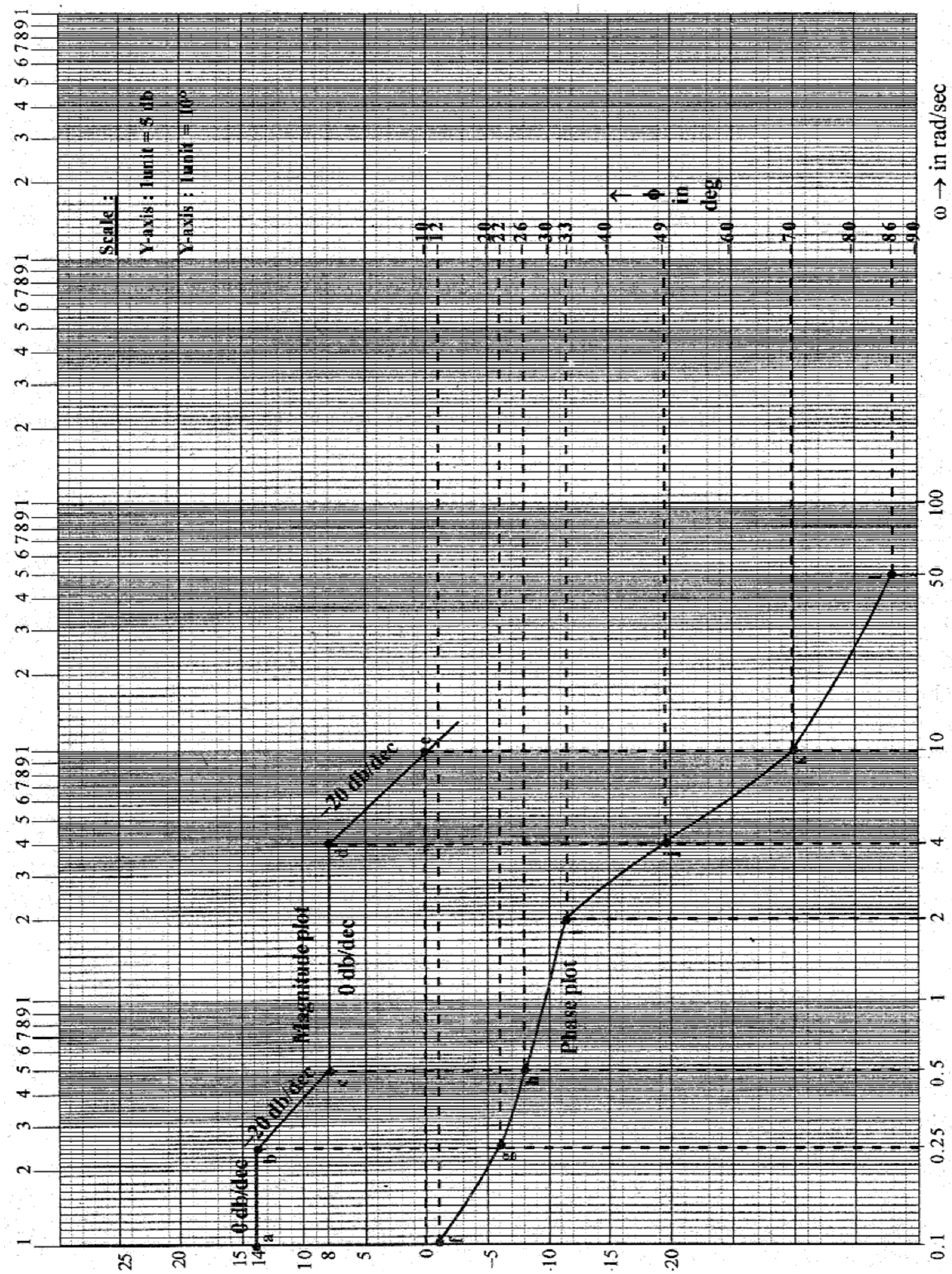


Fig 4.6

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot in a semi log graph sheet choose a scale of 1unit-5db on y-axis. The frequencies are marked in decades from 0.1 to 100 r/s on logarithmic scales in x-axis, Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT:

The phase angle of $G(j\omega)$ as a function of ω is given by,

$$\boldsymbol{\phi} = -\tan^{-1} 2\omega - \tan^{-1} 4\omega - \tan^{-1} 0.25\omega$$

The phase angles of various values of ω are calculated and listed in table 2.

TABLE-2

ω	$\tan^{-1} 2\omega$ deg	$\tan^{-1} 4\omega$ deg	$\tan^{-1} 0.25\omega$ deg	$\phi = \angle G(j\omega)$	Points in phase plot
0.1	11.3	21.8	1.43	$-11.93 \approx -12$	f
0.25	26.56	45.0	3.5	$-21.94 \approx -22$	g
0.5	45.0	63.43	7.1	$-25.53 \approx -26$	h
2	75.96	82.87	26.56	$-33.47 \approx -33$	i
4	82.87	86.42	45.0	$-48.55 \approx -49$	j
10	87.13	88.56	68.19	$-69.62 \approx -70$	k
50	89.42	89.71	85.42	$-85.71 \approx -86$	l

On the same semi log graph sheet, choose a scale of 1 unit=10° on the y-axis on the right side of semilog graph sheet. Mark the phase angles on the graph sheet and join the graph by smooth curve.

4.4 POLAR PLOT

The polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase angle of G(jω) on polar coordinates as ω is varied from zero to infinity. Thus, the polar plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity. The polar plot is also called Nyquist plot.

The polar plot is usually plotted on a polar graph sheet. The polar graph sheet has concentric circles and radial lines. **The circles represent the magnitude and the radial lines represent the phase angles**. Each point on the polar graph has a magnitude and phase angle the magnitude of a point is given by the value of the circle passing through that point and the phase angle is given by the radial line passing through that point. In po1ar graph sheet a positive phase angle is measured in anti-clockwise from the reference axis (0°) and a negative angle is measured clockwise from the reference axis (0°).

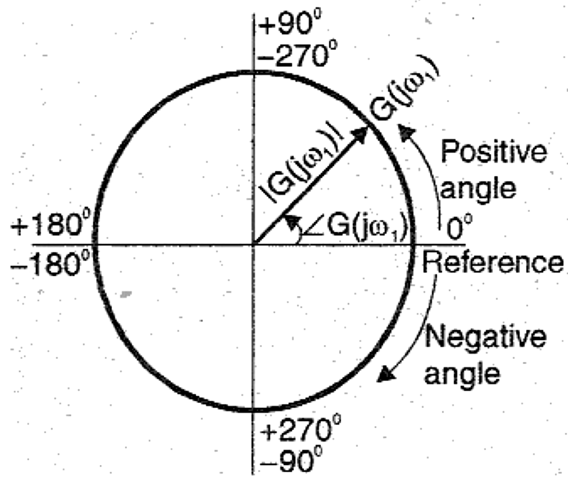


Fig 4.7

To draw the polar plot, magnitude and phase of G(jω) are computed for various values of ω and tabulated.

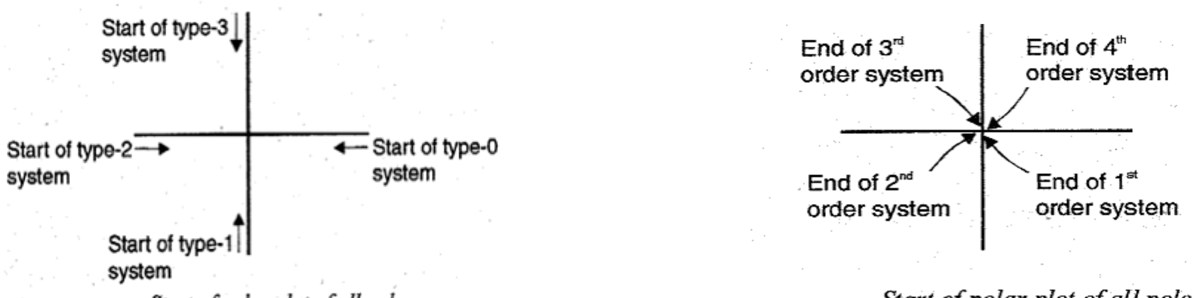
Alternatively, if G(jω) can be expressed in rectangular coordinates as,

$$G(j\omega) = G_R(j\omega) + G_I(j\omega)$$

where, $G_R(j\omega)$ = Real part of G(jω);

$G_I(j\omega)$ = Imaginary part of G(jω).

Then the polar plot can be plotted in ordinary graph sheet between $G_R(j\omega)$ and $G_I(j\omega)$ by varying ω from 0 to ∞. To draw the polar plot on ordinary graph sheet, the magnitude and phase of G(jω) are computed for various values of ω. Then convert the polar coordinates to rectangular coordinates using P→R conversion (polar to rectangular conversion) in the calculator. Sketch the polar plot using rectangular coordinates.



Start of polar plot of all pole

Start of polar plot of all pole minimum phase system.

Fig 4.8

For minimum phase transfer function with only poles, type number of the system determines the quadrant at which the polar plot starts and the order of the system determines the quadrant at which the polar plot ends. The minimum phase systems are systems with all poles and zeros on left half of s- plane. The start and end of polar plot of all pole minimum phase system are shown in figures respectively. Some typical sketches of polar plot are shown in table.

The change in shape of polar plot can be predicted due to addition of a pole or zero.

1. When a pole is added to a system, the polar plot end will shift by -90°.
2. When a zero is added to a system the polar plot end will shift by +90°.

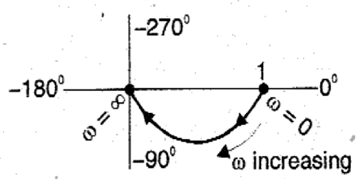
4.4.1 TYPICAL SKETCHES OF POLAR PLOT:

Type : 0, Order : 1 $\qquad G(s)=\frac{1}{1+sT}$

$$G(j\omega)=\frac{1}{1+j\omega T}=\frac{1}{\sqrt{1+\omega^2T^2}\,\angle\tan^{-1}\omega T}=\frac{1}{\sqrt{1+\omega^2T^2}}\angle-\tan^{-1}\omega T$$

As $\omega\to 0,\quad G(j\omega)\to 1\angle 0^\circ$

As $\omega\to\infty,\quad G(j\omega)\to 0\angle-90^\circ$

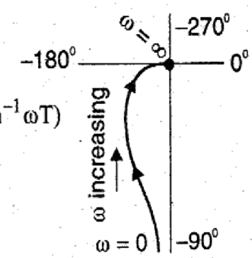


Type : 1, Order : 2 $\qquad G(s)=\frac{1}{s(1+sT)}$

$$G(j\omega)=\frac{1}{j\omega(1+j\omega T)}=\frac{1}{\omega\angle 90^\circ\sqrt{1+\omega^2T^2}\angle\tan^{-1}\omega T}=\frac{1}{\omega\sqrt{1+\omega^2T^2}}\angle(-90^\circ-\tan^{-1}\omega T)$$

As $\omega\to 0,\quad G(j\omega)\to\infty\angle-90^\circ$

As $\omega\to\infty,\quad G(j\omega)\to 0\angle-180^\circ$



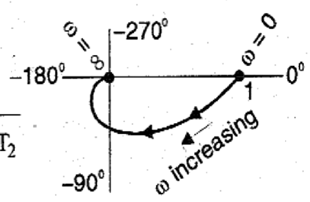
Type : 0, Order : 2 $\qquad G(s)=\frac{1}{(1+sT_1)(1+sT_2)}$

$$G(j\omega)=\frac{1}{(1+j\omega T_1)\,(1+j\omega T_2)}=\frac{1}{\sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\,\sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2}$$

$$=\frac{1}{\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2)$$

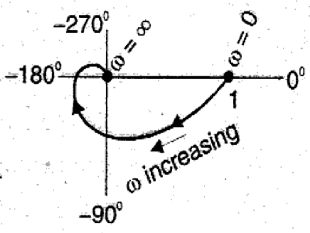
As $\omega\to 0,\quad G(j\omega)\to 1\angle 0^\circ$

As $\omega\to\infty,\quad G(j\omega)\to 0\angle-180^\circ$



Type : 0, Order : 3

$$G(s)=\frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$



$$G(j\omega)=\frac{1}{(1+j\omega T_1)\,(1+j\omega T_2)\,(1+j\omega T_3)}$$

$$=\frac{1}{\sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2\ \sqrt{1+\omega^2T_3^2}\angle\tan^{-1}\omega T_3}$$

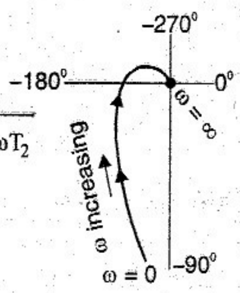
$$=\frac{1}{\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)\left(1+\omega^2T_3^2\right)}}\angle(-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2-\tan^{-1}\omega T_3)$$

As $\omega\to 0$, $\quad G(j\omega)\to 1\angle 0^\circ$

As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle -270^\circ$

Type : 1, Order : 3

$$G(s)=\frac{1}{s(1+sT_1)(1+sT_2)}$$



$$G(j\omega)=\frac{1}{j\omega(1+j\omega T_1)\,(1+j\omega T_2)}=\frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\ \angle\tan^{-1}\omega T_2}$$

$$=\frac{1}{\omega\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-90^\circ-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2)$$

As $\omega\to 0$, $\quad G(j\omega)\to \infty\angle -90^\circ$

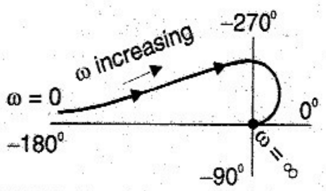
As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle -270^\circ$

Type : 2, Order : 4

$$G(s)=\frac{1}{s^2(1+sT_1)(1+sT_2)}$$

$$G(j\omega)=\frac{1}{(j\omega)^2\ (1+j\omega T_1)\,(1+j\omega T_2)}=\frac{1}{\omega^2\angle-180^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2}$$

$$=\frac{1}{\omega^2\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-180^\circ-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2)$$

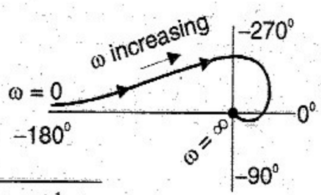


As $\omega\to 0$, $\quad G(j\omega)\to \infty\angle -180^\circ$

As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle -360^\circ$

Type : 2, Order : 5

$$G(s)=\frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$$



$$G(j\omega)=\frac{1}{(j\omega)^2\,(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$=\frac{1}{\omega^2\angle-180^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2\ \sqrt{1+\omega^2T_3^2}\angle\tan^{-1}\omega T_3}$$

$$=\frac{1}{\omega^2\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)\left(1+\omega^2T_3^2\right)}}\angle(-180^\circ-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2-\tan^{-1}\omega T_3)$$

As $\omega\to 0$, $\quad G(j\omega)\to \infty\angle -180^\circ$

As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle -450^\circ = 0\angle -90^\circ$

Type : 1, Order : 1 $G(s) = \frac{1}{s}$

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega\angle 90^\circ} = \frac{1}{\omega}\angle -90^\circ$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -90^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -90^\circ$

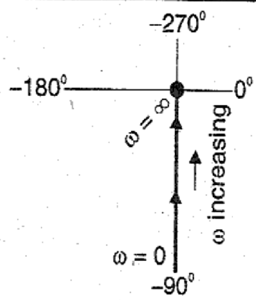


Fig 4.9

4.4.2 DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM POLAR PLOT:

The gain margin is defined as the inverse of the magnitude of $G(j\omega)$ at phase crossover frequency. The phase crossover frequency is the frequency at which the phase of $G(j\omega)$ is 180°.

Let the polar plot cut the 180° axis at point B and the magnitude circle passing through the point B be G_B. Now the Gain margin, $Kg = 1/G_B$. If the point B lies within unity circle, then the Gain margin is positive otherwise negative. (If the polar plot is drawn in ordinary graph sheet using rectangular coordinates then the point B is the cutting point of $G(j\omega)$ locus with negative real axis and $K_g = 1/|G_B|$ where GB is the magnitude corresponding to point B).

The following figure showing positive gain and phase margin.

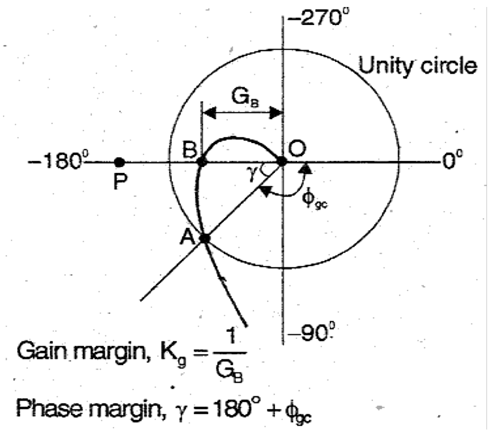


Fig 4.10

The following figure showing negative gain and phase margin.

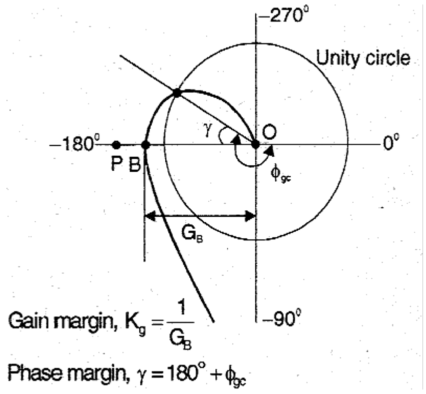


Fig 4.11

The phase margin is defined as, phase margin, γ=180°+ϕ_{gc} where ϕ_{gc} is the phase angle of G(jω) at gain crossover frequency. The gain crossover frequency is the frequency at which the magnitude of G(jω) is unity.

4.4.3 GAIN ADJUSTMENT USING POLAR PLOT:

Draw G(jω) locus with K = 1. Let it cut the -180° axis at point B corresponding to a gain of G. let the specified gain margin be x db. For this gain margin, the G(jω) locus will cut -180° at point A whose magnitude is G_A.

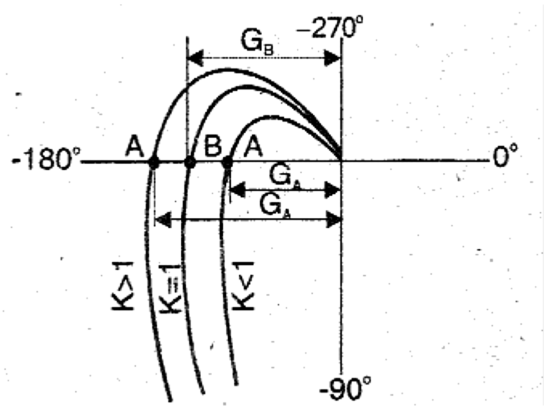


Fig 4.12

Now, $20\log\frac{1}{G_A} = x$

$$\log\frac{1}{G_A} = x/20$$

$$\frac{1}{G_A} = 10^{\frac{x}{20}}$$

$$G_A = \frac{1}{10^{\frac{x}{20}}}$$

Now the value of K is given by, $K = G_A/G_B$

EXAMPLE 4.4

Sketch the polar plots of 1/s.

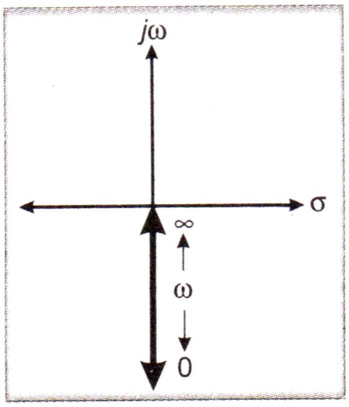


Fig 4.13

SOLUTION

Given that, $G(s) = 1/s$

Puts= jω, we get

$G(j\omega) = 1/j\omega$

If ω=0, then $|G(j\omega)| = \infty$

If ω=∞, then $|G(j\omega)| = 0$

EXAMPLE 4.5

Sketch the polar plots of 1/s (1+sT).

SOLUTION

Given that, $G(s) = 1/s(1+sT)$

Puts= jω, we get

$$G(j\omega) = 1/j\omega(1+j\omega T)$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2 T^2}}$$

$$\angle G(j\omega) = -90° - \tan^{-1} \omega T$$

If ω=0, then $|G(j\omega)| = \infty$ and $\angle G(j\omega) = -90°$

If ω=∞, then $|G(j\omega)| = 0$ and $\angle G(j\omega) = -180°$

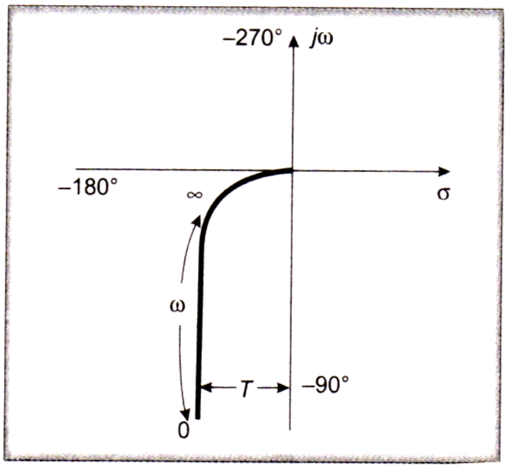


Fig 4.14

EXAMPLE 4.6

The open loop transfer function of a unity feedback system is given by $G(s) = 1/s(1+s)(1+2s)$. Sketch the polar plot and determine the gain margin and phase margin.

SOLUTION

Given that, $G(s) = 1/s(1+s)(1+2s)$

Puts= jω, $G(j\omega) = 1/j\omega(1+j\omega)(1+2j\omega)$

The corner frequencies are $\omega c_1 = 1/2 = 0.5$ rad/sec, $\omega c2 = 1$ rad/sec. The magnitude and phase angle of G(jω) are calculated for the corner frequencies and for frequencies around comer frequencies and tabulated in table. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in figure.

$$G(j\omega) = \frac{1}{j\omega\ (1+j\omega)\ (1+2\,j\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\sqrt{(1+\omega^2)}\ \angle\tan^{-1}\omega\sqrt{(1+4\omega^2)}\angle\tan^{-1}2\omega}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2)}\ \sqrt{(1+4\omega^2)}}\ \angle 90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$|G(j\omega)| \quad = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}}$$

$$= \frac{1}{\omega\sqrt{1+4\omega^2+\omega^2+4\omega^4}}$$

$$|G(j\omega)| \ = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle\, G(j\omega) \quad = -90^\circ\text{-}\tan^{-1}\omega\text{-}\tan^{-1}2\omega$$

Table:

Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
|G(jω)|	2.2	1.8	1.5	1.2	0.9	0.7	0.3
∠G(jω) deg	-144	-150	-156	-162	-171	-179.5 ≈-180	-198

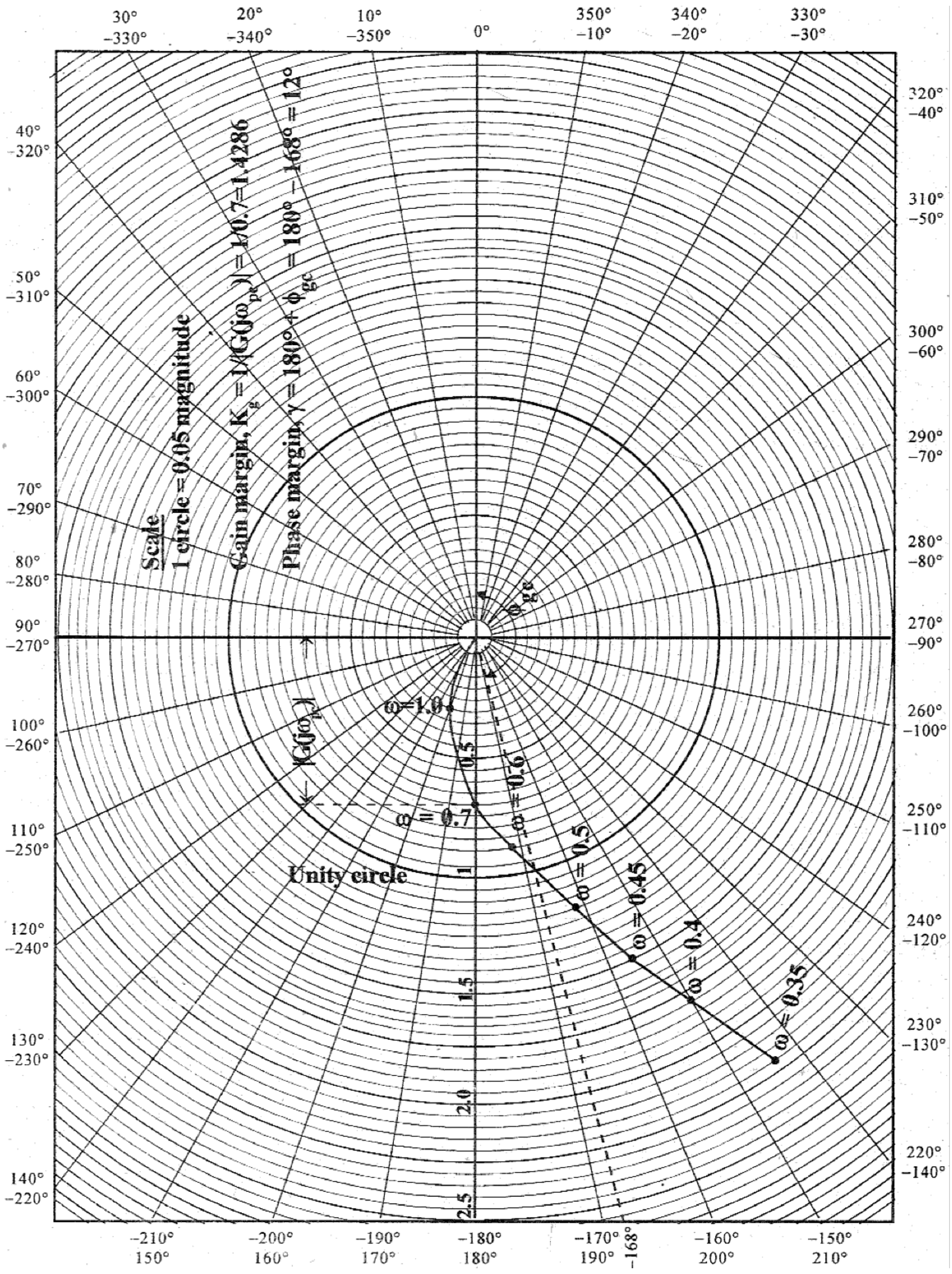


Fig 4.15

RESULT:

1. Gain margin, $K_g = 1.42$
2. Phase margin, $\gamma = +12°$

EXAMPLE 4.7

The open loop transfer function of a unity feedback system is given by $G(s) = 1/s(1+s)^2$. Sketch the polar plot and determine the gain margin and phase margin.

SOLUTION

Given that, $G(s) = 1/s\,(1+s)^2$

Puts= jω

$G(j\omega) = 1/\,j\omega\,(1+j\omega)^2$

The corner frequency ωc1= 1rad/sec. The magnitude and phase angle of G (jω) are calculated for the corner frequencies and for frequencies around comer frequencies and tabulated in table. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in figure.

$$G(j\omega) = \frac{1}{j\omega\,(1+j\omega)^2}$$

$$= \frac{1}{\omega\angle 90°\sqrt{(1+\omega^2)}\,\angle\tan^{-1}\omega\sqrt{(1+\omega^2)}\,\angle\tan^{-1}\omega}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2)}\,\sqrt{(1+\omega^2)}}\angle 90° - \tan^{-1}\omega - \tan^{-1}\omega$$

$$|G(j\omega)| = \frac{1}{\omega(\sqrt{(1+\omega^2)})^2}$$

$$= \frac{1}{\omega(1+\omega^2)}$$

$$|G(j\omega)| = \frac{1}{\omega+\omega^3}$$

$$\angle G(j\omega) = -90° - 2\tan^{-1}\omega$$

Table:

Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
|G(jω)|	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4
∠G(jω) deg	−134	−143	−151	−159	−167	−174	−180	−185

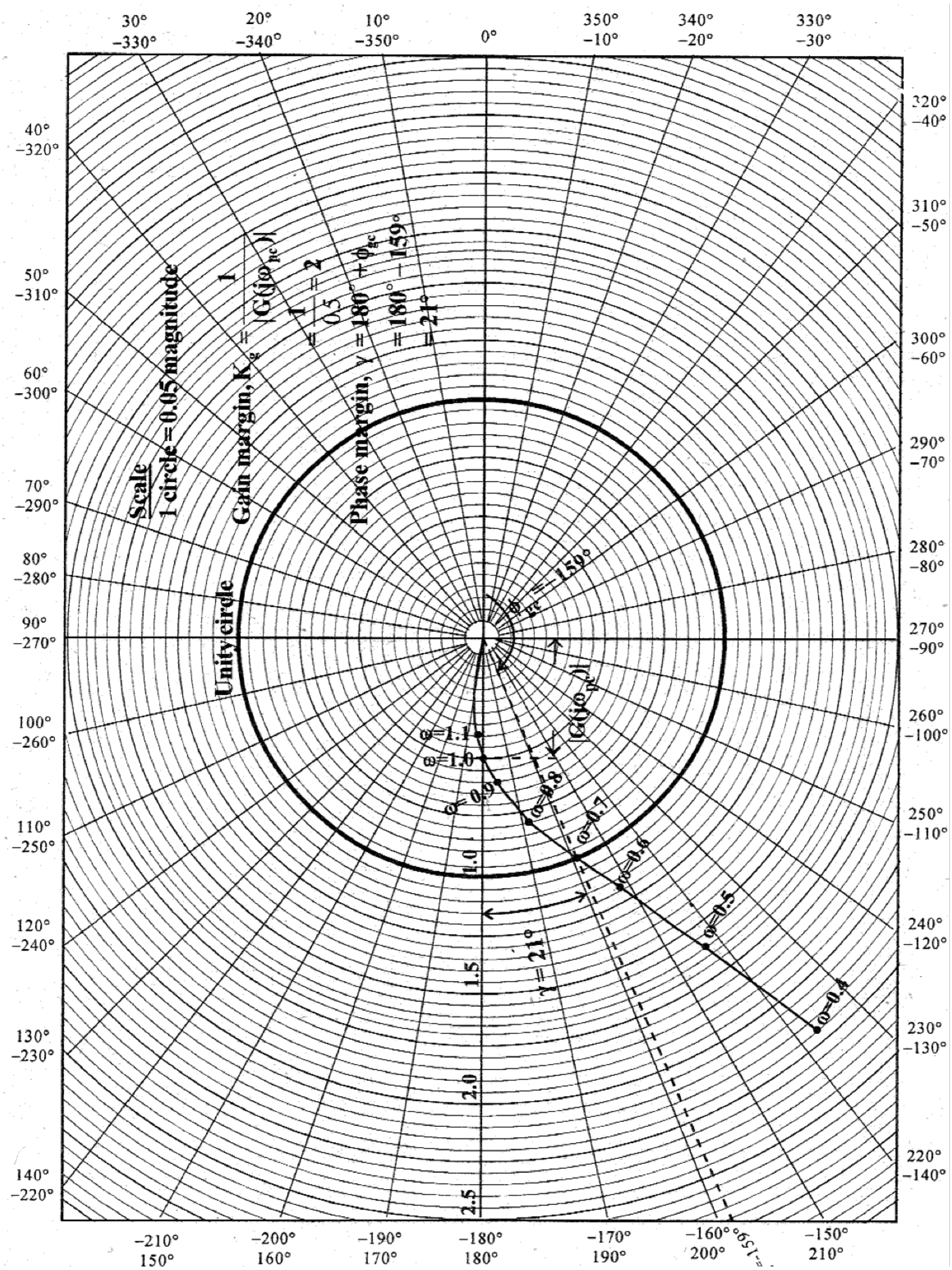


Fig 4.16

RESULT:

1. Gain margin, $K_g = 2$
2. Phase margin, $\gamma = -21°$

EXAMPLE 4.8

The open loop transfer function of a unity feedback system is given by $G(s) = 1/s(1+0.5s)(1+4s)$. Sketch the polar plot and determine the value of K so that (i) gain margin is 20db and (ii) phase margin is 30°.

SOLUTION

Given that, $G(s) = 1/s(1+0.5s)(1+4s)$

The polar plot is sketched by taking K=1.

Puts= jω in G(s),

$$G(j\omega) = 1/j\omega(1+0.5j\omega)(1+4j\omega)$$

The corner frequency $\omega c_1 = 1/4 = 0.25$ rad/sec and $\omega c_2 = 1/0.5 = 2$ rad/sec. The magnitude and phase angle of G(jω) are calculated for the corner frequencies and for frequencies around comer frequencies and tabulated in table. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in figure.

$$G(j\omega) = \frac{1}{j\omega\,(1+0.5j\omega)\,(1+4j\omega)}$$

$$= \frac{1}{\omega\angle 90°\sqrt{(1+(0.5\omega)^2)}\,\angle\tan^{-1}0.5\omega\sqrt{(1+(4\omega)^2)}\angle\tan^{-1}4\omega}$$

$$= \frac{1}{\omega\sqrt{(1+0.25\omega^2)}\,\sqrt{(1+16\omega^2)}}\angle 90° - \angle\tan^{-1}0.5\omega - \tan^{-1}4\omega$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{(1+0.25\omega^2)(1+16\omega^2)}}$$

$$\angle G(j\omega) = -90° - \tan^{-1}0.5\,\omega - \tan^{-1}4\omega$$

Table:

Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
|G(jω)|	2.11	1.3	0.87	0.61	0.35	0.22	0.15
∠G(jω) deg	−149	−159	−167	−174	−184	−193	−199

From the polar plot,

(i). Gain margin $K_g = 2.27$, in db $K_g = 7.12$db.

(ii). Phase margin $\gamma = 15°$.

Case(i):

With K=1, let the G(jω) cut the -180° axis at the point B and gain corresponding to that point be G_B. From the polar plot, $G_B = 0.44$.

The gain margin of 7.12db with K=1 has be increased to 20db and so K must be decreased to a value less than 1.

Let G_A be the gain at -180° for a gain margin of 20db.

Now $20\log(1/G_A) = 20$

$\text{Log}\ (1/G_A) = 20/20 = 1$

$$\frac{1}{G_A} = 10^1 = 10$$

$$G_A = 1/10 = 0.1$$

Then the value of $K = G_A/G_B = 0.1/0.44 = 0.227$

Case(ii):

With K=1, let the phase margin is 15°. This must be increased to 30°.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 30°

$$30° = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 30° - 180° = -150°$$

In the polar plot the -150° line cuts the locus of G(jω) at point A and cut the unity circle at point B.

Let G_A = magnitude of G(jω) at point A.

G_B = magnitude of G(jω) at point B.

From the polar plot, $G_A = 2.04$ and $G_B = 1$

Then the value of $K = G_B/G_A = 1/2.04 = 0.49$

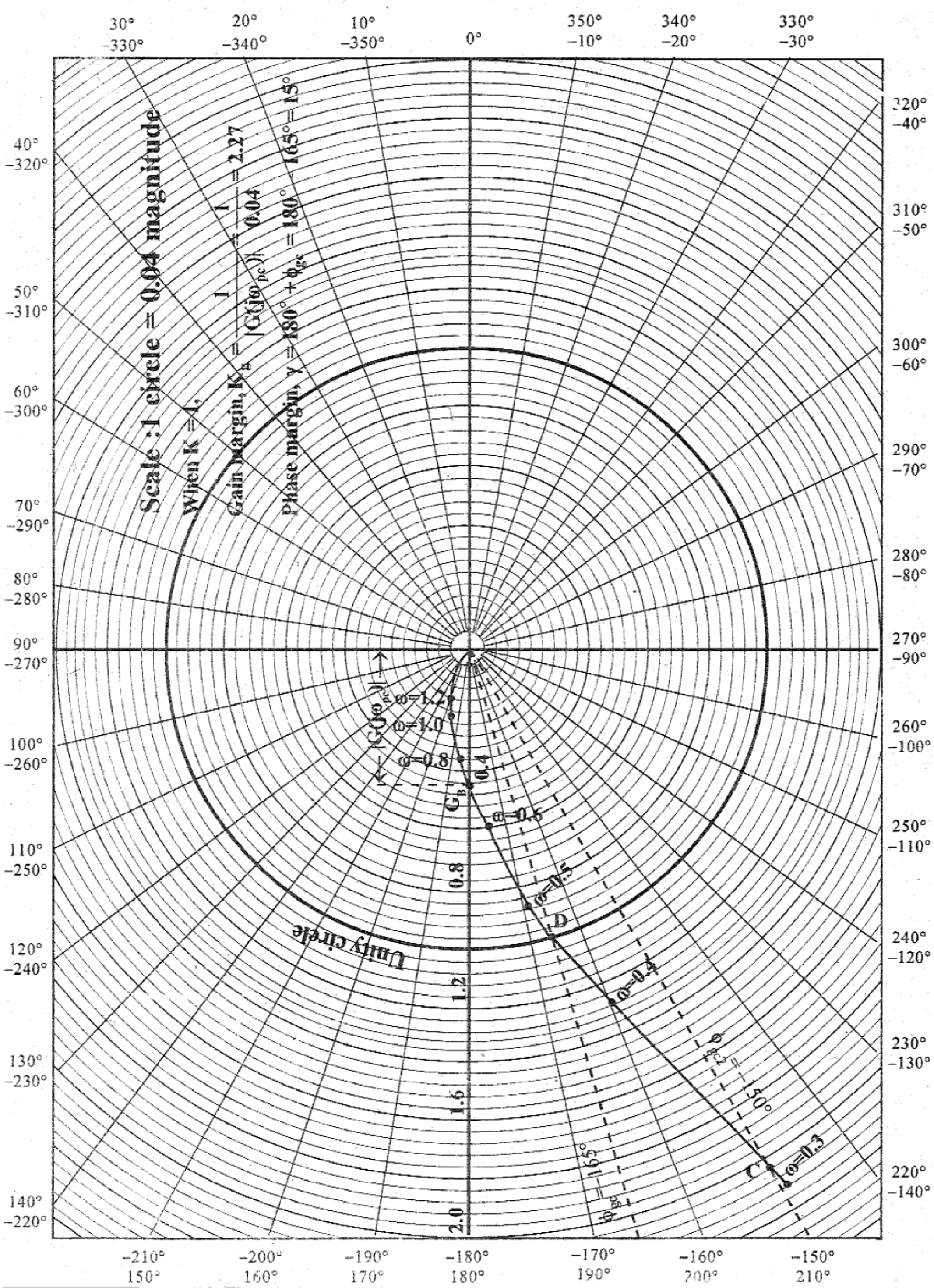


Fig 4.17

RESULT:

1. Gain margin K_g = 2.27, in db K_g = 7.12db.
2. Phase margin γ = 15°.
3. For gain margin of 20db, K=0.227
4. For phase margin of 30°, K=0.49

REVIEW QUESTIONS

PART A

1. What is frequency response?
2. List the advantages of frequency response analysis.
3. List the frequency domain specifications.
4. What is resonant peak?
5. Define resonant frequency.
6. What is gain cross over frequency?
7. What is phase cross over frequency?
8. What is Bode plot?
9. What is Polar plot?
10. Mention the type number and order of $G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$

PART B

1. Explain gain margin and phase margin.
2. Draw the Bode plot of $G(s) = \frac{1}{s(1+sT)}$
3. How can you determine gain margin and phase margin in polar plot?
4. Draw the polar plot of $G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$
5. Draw the polar plot of $G(s) = 1/s^2$

PART C

1. Explain the procedure for constructing Bode plot.
2. Sketch Bode plot for the following transfer function and determine gain cross over frequency and phase cross over frequency. $G(s) = \frac{20}{S(1+3S)(1+4S)}$
3. Sketch Bode plot for the following transfer function and determine gain cross over frequency and phase cross over frequency. $G(s) = \frac{1}{S(1+0.5S)(1+0.2S)}$
4. Sketch Bode plot for the following transfer function and determine gain cross over frequency and phase cross over frequency. $G(s) = \frac{10(s+10)}{s(s+2)(s+5)}$
5. The open loop transfer function of a unity feedback system is given by $G(s) = 1/s^2(1+s)\ (1+2s)$. Sketch the polar plot and determine the gain margin and phase margin.

UNIT V

STABILITY

5.1 STABILITY

5.1.1 DEFINITIONS OF STABIITY

The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system, the response or output is predictable, finite and stable for a given input (or for any changes in input or for any changes in system parameters).

The different definitions of the stability are the following.

Stable system:

"A system is stable, if its output is bounded (finite) for any bounded (finite) input."

Asymptotically stable system:

"A system is asymptotically stable, if in the absence of the input the output tends towards zero (or to the equilibrium state) irrespective of initial conditions"

Unstable system:

"A system is unstable if for a bounded disturbing input signal the output is of infinite amplitude or oscillatory"

Limitedly stable system:

"For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called limitedly stable system"

Absolutely stable system:

"If a system output is stable for all variations of its parameters, then the system is called absolutely stable system"

Conditionally stable system:

"If a system output is stable for a limited range of variations of its parameters, then the system is called conditionally stable system"

BIBO stability:

"A linear relaxed system is said to have BlBO stability if every bounded (finite) input results in a bounded (finite) output"

Relative stability:

"The Relative stability indicates the closeness of the system to stable region. It is an indication of the strength or degree of stability"

RESPONSE OF A SYSTEM

Let the Closed loop transfer function $\frac{C(S)}{R(S)} = M(s)$

The response or output in s domain C(s)= M(s)R(s)

C(s) = Output in s-domain.

R(s) = Input in s-domain.

5.2 LOCATION OF POLES ON s-PLANE FOR STABILITY

The closed loop transfer function, M(s) can be expressed as a ratio of two polynomials in s. The Denominator polynomial of closed loop transfer function is equated to zero, the equation obtained is called as characteristic equation. The roots of characteristic equation are poles of closed loop transfer function.

For BIBO stability the integral of impulse response should be finite, which implies that the impulse response should be finite as t tends to infinity. [The impulse response is the inverse Laplace transform of the transfer function]. This requirement for stability can be linked to the location of roots of the characteristic equation in the s-plane.

The closed loop transfer function M(s) can be expressed as a ratio of two polynomials,

$$M(s) = \frac{(s+z_1)\,(s+z_2)\,(s+z_3)....(s+z_m)}{(s+p_1)\,(s+p_2)\,(s+p_3)....(s+p_n)}$$

The roots of numerator polynomial $z_1,\ z_2, z_3,\ .....\ z_n$ are zeros.

The roots of denominator polynomial $p_1,\ p_2, p_3 \ldots p_m$ are poles.

The denominator polynomial gives the characteristic equation and so the poles are roots of characteristic equation.

From table, the following conclusions are drawn based on the location of roots of characteristic equation.

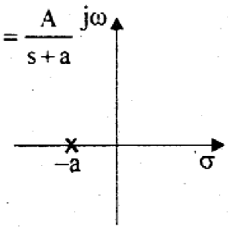
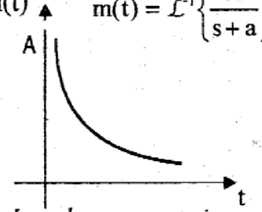
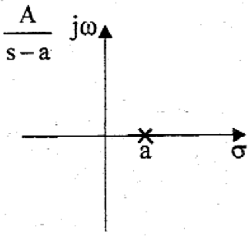
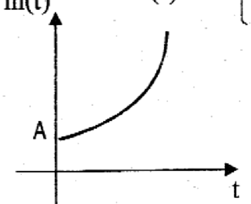
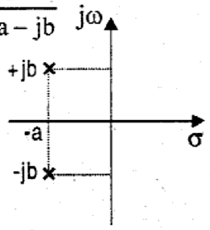
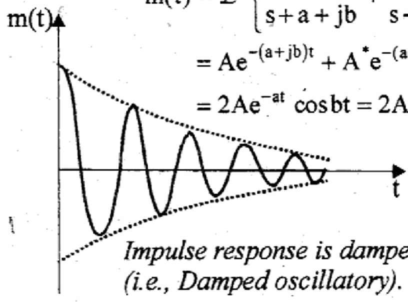
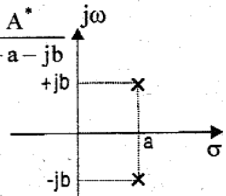
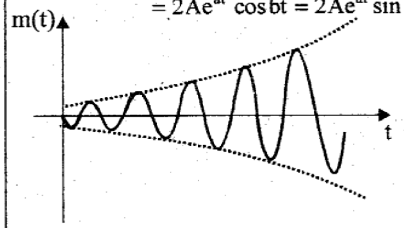
1. If all the roots of characteristic equation have negative real parts (i.e., lying on left half s-plane) then the impulse response is bounded (Le.,. it decreases to zero, as t tends to ∞).

Hence, the system has bounded-input and bounded-output so the system is stable.

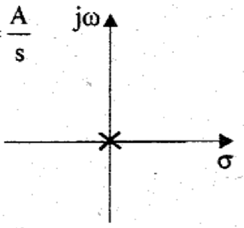
2. If any root of the characteristic equation has a positive real part (i.e., lying on right half s-plane) then impulse response is unbounded, (i.e., it increases to ∞ as t tends to ∞).

Hence the system is unstable.

3. If the characteristic equation has repeated roots on the imaginary axis then impulse response is unbounded (i.e., it increases to ∞ as t tends to ∞). Hence the system is unstable.

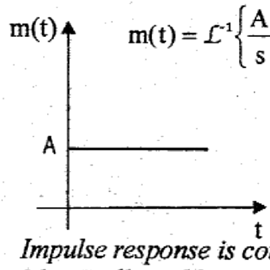
Transfer function, M(s) and location of roots on s-plane	Impulse response, m(t)
$M(s) = \frac{A}{s+a}$  *Root on negative real axis*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+a}\right\} = Ae^{-at}$  *Impulse response is exponentially decaying. Stable system.*
$M(s) = \frac{A}{s-a}$  *Root on positive real axis*	$M(s) = \mathcal{L}^{-1}\left\{\frac{A}{s-a}\right\} = Ae^{+at}$  *Impulse response is exponentially increasing. Unstable system.*
$M(s) = \frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}$  *Complex conjugate roots on left half of s-plane*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}\right\}$ $= Ae^{-(a+jb)t} + A^*e^{-(a-jb)t}$ $= 2Ae^{-at}\cos bt = 2Ae^{-at}\sin\left(bt+90^\circ\right)$  *Impulse response is damped sinusoidal (i.e., Damped oscillatory). Stable system*
$M(s) = \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}$  *Complex conjugate roots on right half of s-plane*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}\right\}$ $= Ae^{-(-a+jb)t} + A^*e^{-(-a-jb)t}$ $= 2Ae^{at}\cos bt = 2Ae^{at}\sin(bt+90^\circ)$  *Impulse response is exponentially increasing sinusoidal (i.e., Amplitude of oscillations exponentially increases with time). Unstable system.*

$$M(s) = \frac{A}{s}$$



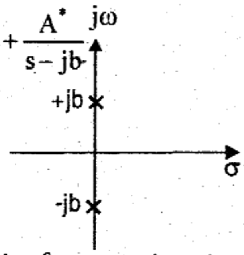
Single root at origin

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s}\right\} = A$$



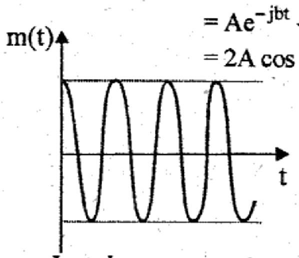
Impulse response is constant.
Marginally stable system.

$$M(s) = \frac{A}{s+jb} + \frac{A^*}{s-jb}$$



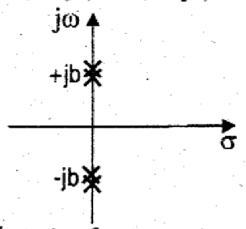
Single pair of roots on imaginary axis

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+jb} + \frac{A^*}{s-jb}\right\}$$
$$= Ae^{-jbt} + A^*e^{+jbt}$$
$$= 2A\cos bt = 2A\sin(bt + 90°)$$



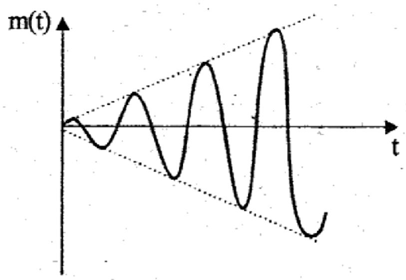
Impulse response is oscillatory
Marginally stable

$$M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$$



Double pair of roots on imaginary axis

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}\right\}$$
$$= At\,e^{-jbt} + A^*\,t\,e^{+jbt}$$
$$= 2At\cos bt = 2At\sin(bt + 90°)$$



Impulse response is linearly increasing sinusoidal (i.e., amplitude of oscillations linearly increases with time). Unstable system.

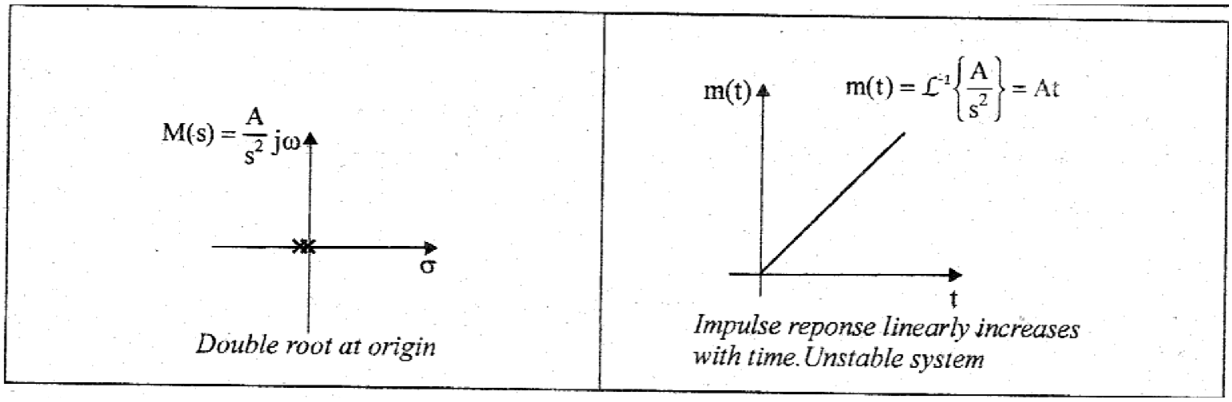


Fig 5.1

4. If one or more non - repeated roots of the characteristic equation are lying on the imaginary axis, then impulse response is bounded (i.e., it has constant amplitude oscillations) hence the system-is unstable.

5. If the characteristic equation has single root at origin then the impulse response is bounded (i.e., it has constant amplitude) hence the system is unstable.

6. If the characteristic equation has repeated roots at origin then the impulse response is unbounded (i.e., it linearly increases to infinity as t tends to ∞) and so the system is unstable.

7. In system with one or more non-repeated roots on imaginary axis or with single root at origin, the output is bounded for bounded inputs except for the inputs having poles matching the system poles. These cases may be treated as acceptable or non-acceptable. Hence when the system has non-repeated poles on imaginary axis or single pole at origin, it is referred as limitedly or marginally stable system.

In summary, the following three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation.

1. If all the roots of characteristic equation have negative real parts, then the system is stable.

2. If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis then the system is unstable.

3. If the condition (i) is satisfied except for the presence of one or more non-repeated roots on the imaginary axis, then the system is limitedly or marginally stable.

In summary, the following conclusions can be made about coefficients of characteristic polynomial.

1. If all the coefficients are positive and if no coefficient is zero, then all the roots are in the left half of s- plane.

2. If any coefficient a_i is equal to zero then, some of the roots may be on the Imaginary axis or the right half of s- plane.

3. If any coefficient a_i is negative, then at least one root is in the right half of $- plane.

5.3 ROUTH HURWITZ CRITERION

The Routh-Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial have negative real part or not. '

The first step in analyzing the stability of a system is to examine its characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial be positive. If some of the coefficients are zero or negative it can be concluded that the system is unstable.

When all the coefficients are positive, the system is not necessarily stable. Even though the coefficients are positive, some of the roots may lie on the right half of s-plane or on the imaginary axis. For all the roots to have negative real parts, it is necessary but not sufficient that all coefficients of the characteristic equation be positive. If all the coefficients of the characteristic equation are positive, then the system may be stable and one should proceed further to examine the sufficient conditions of stability.

A. Hurwitz and E.J. Routh independently published the method of investigating the sufficient conditions of stability of a system. The Hurwitz criterion is in terms of determinants and Routh criterion is in terms of array formulation. The Routh stability criterion is presented here.

The Routh stability criterion is based on ordering the coefficients of the characteristic equation,

$a_0s^n+a_1s^{n-1}+a_2s^{n-2}+\ldots+a_{n-1}s+a_n=0$, where $a_0>0$ into a schedule, called the Routh array as shown below.

$$\begin{array}{lcllllll}
s^n & : & a_0 & a_2 & a_4 & a_6 & a_8 & \ldots\ldots \\
s^{n-1} & : & a_1 & a_3 & a_5 & a_7 & a_9 & \ldots\ldots \\
s^{n-2} & : & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots\ldots \\
s^{n-3} & : & c_0 & c_1 & c_2 & c_3 & c_4 & \ldots\ldots \\
 & & & & & & & \\
s^1 & : & g_0 & & & & & \\
s_0 & : & h_0 & & & & &
\end{array}$$

The Routh stability criterion can be stated as follows.

"**The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive.** If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane.

Note: If the order of sign of first column element is +, +, -, + and +. Then + to - is considered as one sign change and - to + as another sign change.

5.3.1 CONSTRUCTION OF ROUTH ARRAY

Let the characteristic polynomial be,

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n = 0$$

The coefficients of the polynomial are arranged in two rows as shown below.

$$\begin{array}{lcll}
s^n & : & a_o\ a_2\ a_4 & a_6 \ldots. \\
s^{n-1} & : & a_1\ a_3\ a_5 & a_7 \ldots.
\end{array}$$

When n is even, the s^n row is formed by coefficients-of even order terms (i.e., co-efficient of even powers of s) and s^{n-1} row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s).

When n is odd, the s^n row is formed by coefficients of odd order terms (i.e., co-efficients of odd powers of s) and s^{n-1} row is formed by coefficients of even order terms (i.e., coefficients of even powers of s).

The other rows of Routh array up to s^0 row can be formed by the following procedure. Each row of the Routh array is constructed by using the elements of previous two rows.

Consider two consecutive rows of Routh array as shown below,

$$\begin{array}{llllllll} s^{n-x} & : & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \ldots\ldots \\ s^{n-x-1} & : & y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \ldots\ldots \end{array}$$

Let the next row be,

$$s^{n-x-2} : z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \ldots\ldots$$

The elements of s^{n-x-2} row are given by,

$$z_0 = \frac{(-1)\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}}{y_0} = \frac{y_0x_1 - y_1x_0}{y_0}$$

$$z_1 = \frac{(-1)\begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}}{y_0} = \frac{y_0x_2 - y_2x_0}{y_0}$$

$$z_2 = \frac{(-1)\begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix}}{y_0} = \frac{y_0x_3 - y_3x_0}{y_0}$$

$$z_3 = \frac{(-1)\begin{vmatrix} x_0 & x_4 \\ y_0 & y_4 \end{vmatrix}}{y_0} = \frac{y_0x_4 - y_4x_0}{y_0}$$

$$z_4 = \frac{(-1)\begin{vmatrix} x_0 & x_5 \\ y_0 & y_5 \end{vmatrix}}{y_0} = \frac{y_0x_5 - y_5x_0}{y_0} \quad \text{and so on.}$$

The elements z_0, z_1, z_2, z_3,..... are computed for all possible computations as shown above.

In the process of constructing Routh array the missing terms are considered as zeros. Also, all the elements of any row can be multiplied or divided by a Positive constant to simplify the computational work.

In the construction of Routh array one may come across the following three cases.

Case-I: Normal Routh array (Non-zero elements in the first column of Routh array).

Case-I: A row of all zeros.

Case-III: First element of a row is zero but some or other elements are not zero.

Case-I: Normal Routh Array

In this case, there is no difficulty in forming Routh array. The Routh array can be constructed as explained above. The sign changes are noted to find the number of roots lying on the right half of s-plane and the stability of the system can be estimated.

In this case,

1. If there is no sign change in the first column of Routh array then all the roots are lying on left half of s-plane and the system is stable.

2. If there is sign change in the first column of Routh array, then the system is unstable and the number of roots lying on the right half of s-plane is equal to number of sign changes. The remaining roots are lying on the left half of s-plane.

Case-II: A row of all zeros

An all zero row indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial, the exponents of s are even integers or zero only. This even polynomial factor is also called auxiliary polynomial. The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in the array.

The roots of an even polynomial occur in pairs that are equal in magnitude and opposite in sign. Hence, these roots can be purely imaginary, purely real or complex. The purely imaginary and purely real roots occur in pairs. The complex roots occur in groups of four and the complex roots have quadrant symmetry, that is the roots are symmetrical with respect to both the real and imaginary axes. The following figure shows the roots of an even polynomial.

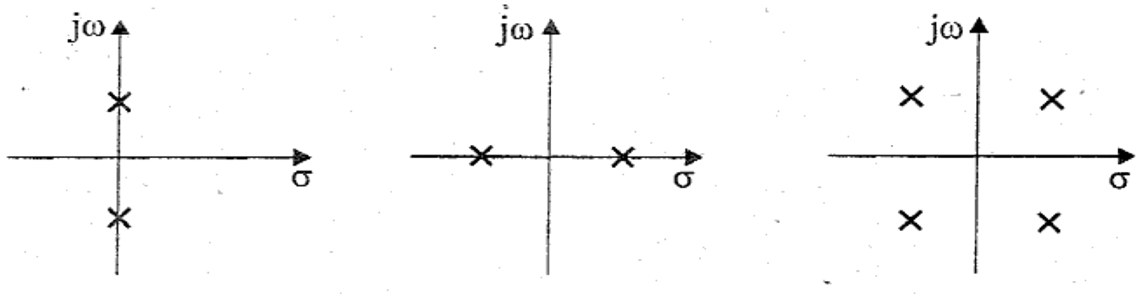


Fig 5.2

The case-II polynomial can be analyzed by anyone of the following two methods.

METHOD: l

1. Determine the auxiliary polynomial, A(s)

2. Differentiate the auxiliary polynomial with respect to s, to get dA(s)/ds

3. The row of zeros is replaced with coefficients of dA(s)/ds.

4. Continue the construction of the array in the usual manner (as that of ease-L) and the array is interpreted as follows.

a. If there are sign changes in the first column of Routh array, then the system is unstable.

The number of roots-lying on right half of s-plane is equal to number of sign changes. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

b. If there are no sign changes in the first column of Routh array then the all zeros row indicates the existence of purely imaginary roots and so the system is limitedly or marginally stable. The roots of auxiliary equation lie on imaginary axis and the remaining roots lies on left half of s-plane,

METHOD-2

1. Determine the auxiliary polynomial, A(s).

2. Divide the characteristic equation by auxiliary polynomial.

3. Construct Routh array using the coefficients of quotient polynomial.

4. The array is interpreted as follows.

a. If there are sign changes in the first column of Routh array of quotient polynomial then the system is unstable. The number of roots of quotient polynomial lying on right half of s- plane is given by number of sign changes in first column of Routh array.

The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex.

The total number of roots on right half of s-plane is given by the sum of number of sign changes and the number of roots of auxiliary polynomial with positive real part. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane,

b. If there is no sign change in the first column of Routh array of quotient polynomial then the system is limitedly or marginally stable. Since there is no sign change all the roots of quotient polynomial are lying on the left half of s-plane.

The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex. The number of roots lying on imaginary axis and on the right half of s-plane can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

Case-III: First element of a row is zero

While constructing Routh array, if a zero is encountered as first element of a row then all the elements of the next row will be infinite. To overcome this problem let $0 \rightarrow \varepsilon$ and complete the construction of array in the usual way (as that of case-I)

Finally let $€ \rightarrow 0$ and determine the values of the elements of the array which are functions of €.

The resultant array is interpreted as follows.

Note: If all the elements of a row are zeros then the solution is attempted by considering the polynomial as case-Il polynomial. Even if there is a single element zero on S^1 row, it is considered as a row of all zeros.

a. If there is no sign change in first column of Routh array and if there is no row with a zero, then all the roots are lying on left half of s-plane and the system is stable.
b. If there are sign changes in first column of Routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of toots lying on the right half of s-plane is equal to number of sign changes and the remaining roots are lying on the left half of s-plane.
c. If there is a row of all zeros after letting $0 \rightarrow \varepsilon$ then there is a possibility of roots on imaginary axis. Determine the auxiliary polynomial and divide the characteristic equation by auxiliary polynomial to eliminate the imaginary roots. The Routh array is constructed using the coefficients of quotient polynomial and the characteristic equation is interpreted as explained in method-2 of case-II polynomial.

EXAMPLE 5.1

Using Routh criterion, determine the stability of the system represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^4+8s^3+18s^2+16s+5=0$

The given characteristic equation is 4^{th} order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of Routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4	:	1	18	5	....Row-1
s^3	:	8	16		....Row-2

The elements of s^3 row can be divided by 8 to simplify the computations.

		Column-1			
s^4	:	1	18	5	....Row-1
s^3	:	1	2		....Row-2
s^2	:	16	5		....Row-3
s^1	:	1.7			....Row-4
s^0	:	5			....Row-5

$$s^2 : \frac{1\times 18-2\times 1}{1} \quad \frac{1\times 5-0\times 1}{1}$$

$$s^2 : 16 \quad 5$$

$$s^1 : \frac{16\times 2-5\times 1}{16}$$

$$s^1 : 1.6875\approx 1.7$$

$$s^0 : \frac{1.7\times 5-0\times 16}{17}$$

$$s^0 : 5$$

On examining the elements of first column of Routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.

RESULT

1. Stable system
2. All the four roots are lying on the left half of s-plane.

EXAMPLE 5.2

Construct Routh array and determine the stability of the system whose characteristic equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

SOLUTION

The characteristic equation of the system is, $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$.

The given characteristic polynomial is 6^{th} order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of Routh array using the coefficients of even powers of S and form the second row using the coefficients of odd powers of s.

s^6	:	1	8	20	16	....Row-1
s^5	:	2	12	16		....Row-2

The elements of s^5 row can be divided by 2 to simplify the calculations.

s^6	:	1	8	20	16	.... Row-1
s^5	:	1	6	8		....Row-2
s^4	:	1	6	8		....Row-4
s^3	:	0	0			....Row-4
s^3	:	1	3			....Row-4
s^2	:	3	8			....Row-5
s^1	:	0.33				....Row-6
s^0	:	8				....Row-7

Column-1

$$s^4 : \frac{1\times 8-6\times 1}{1} \quad \frac{1\times 20-8\times 1}{1} \quad \frac{1\times 16-0\times 1}{1}$$

$$s^4 : \quad 2 \quad 12 \quad 16$$

divide by 2

$$s^4 : \quad 1 \quad 6 \quad 8$$

$$s^3 : \frac{1\times 6-6\times 1}{1} \quad \frac{1\times 8-8\times 1}{1}$$

$$s^3 : \quad 0 \quad 0$$

The auxiliary equation is, $A = s^4+6s^2+8$. On differentiating A with respect to s we get,

$$\frac{dA}{ds} = 4s^3 + 12s$$

The coefficients of $\frac{dA}{ds}$ are used to form s^3 row.

$$s^3 : \quad 4 \quad 12$$

divide by 4

$$s^3 : \quad 1 \quad 3$$

$$s^2 : \frac{1\times 6-3\times 1}{1} \quad \frac{1\times 8-0\times 1}{1}$$

$$s^2 : \quad 3 \quad 8$$

$$s^1 : \frac{3\times 3-8\times 1}{3}$$

$$s^1 : \quad 0.33$$

$$s^0 : \frac{0.33\times 8-0\times 3}{0.33}$$

$$s^0 : \quad 8$$

On examining the elements of 1st column of Routh array it is observed that there is no sigh change. The row with all zeros indicates the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable,

The auxiliary polynomial is,

$$s^4 + 6s^2 + 8 = 0$$

Let, $s^2 = X$

$$x^2+6x+8=0$$

The roots of quadratic are, $x = \frac{-6\pm\sqrt{36-4\times 8}}{2}$

$$= -3 \pm 1$$

$$x = -2 \text{ or } -4$$

The roots of auxiliary polynomial are $s=\sqrt{x} = \pm\sqrt{-2}$ and $\pm\sqrt{-4}$

$$=+j\sqrt{2},\ -j\sqrt{2},\ +j2 \text{ and } -j2$$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Four roots are lying on imaginary axis and remaining two roots are lying on left half of s-plane.

EXAMPLE 5.3

Construct Routh array and determine the stability of the, system represented by the characteristic equation, $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^5+s^4+2s^3+2s^2+3s+5=0$.

The given characteristic polynomial is 5^{th} order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of Routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$
\begin{array}{llllll}
s^5 & : & 1 & 2 & 3 & \text{.... Row-1} \\
s^4 & : & 1 & 2 & 5 & \text{.... Row-2} \\
s^3 & : & \in & -2 & & \text{.... Row-3} \\
s^2 & : & \dfrac{2\in+2}{\in} & 5 & & \text{.... Row-4} \\
s^1 & : & \dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2} & & & \text{.... Row-5} \\
s^0 & : & 5 & & & \text{.... Row-6}
\end{array}
$$

$$
\begin{array}{l}
s^3: \dfrac{1\times 2-2\times 1}{1} \quad \dfrac{1\times 3-5\times 1}{1} \\
s^3: \quad 0 \qquad -2 \\
\text{Replace 0 by } \in \\
s^3: \ \in \qquad -2
\end{array}
$$

$$
\begin{array}{l}
s^2: \dfrac{\in\times 2-(-2\times 1)}{\in} \quad \dfrac{\in\times 5-0\times 1}{\in} \\
s^2: \dfrac{2\in+2}{\in} \qquad 5
\end{array}
$$

$$
\begin{array}{l}
s^1: \dfrac{\dfrac{2\in+2}{\in}\times(-2)-(5\times\in)}{\dfrac{2\in+2}{\in}} \\
s^1: \dfrac{-(5\in^2+4\in+4)}{2\in+2}
\end{array}
$$

On letting $\in \to 0$, we get

$$
\begin{array}{llllll}
s^5 & : & 1 & 2 & 3 & \text{.... Row-1} \\
s^4 & : & 1 & 2 & 5 & \text{.... Row-2} \\
s^3 & : & 0 & -2 & & \text{.... Row-3} \\
s^2 & : & \infty & 5 & & \text{.... Row-4} \\
s^1 & : & -2 & & & \text{.... Row-5} \\
s^0 & : & 5 & & & \text{.... Row-6}
\end{array}
$$

+ to −

− to +

Column-1

$$
\begin{array}{l}
s^0: \dfrac{\dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2}\times 5-0\times\dfrac{2\in+2}{\in}}{\dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2}} \\
s^0: 5
\end{array}
$$

On observing the elements of first column of Routh array, it is found that there are two sign changes. Hence two roots are lying on the right half of s- plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

RESULT

1. The system is unstable.
2. Two roots are lying on right half of s-plane and three roots are lying on left half of s-plane.

EXAMPLE 5.4

By Routh stability criterion determine the stability of the system represented by the characteristic equation, $9s^5-20s^4+10s^3-s^2-9s-10=0$. Comment on the location of roots of characteristic equation.

SOLUTION

The characteristic polynomial of the system is, $9s^5-20s^4+10s^3-s^2-9s-10=0$.

On examining the coefficients of the characteristic polynomial, it is found that some of the coefficients are negative and so some roots will lie on the right half of-s-plane. Hence the system in is unstable. The Routh array can be constructed to find the number of roots lying on right half of s-plane.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of Routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of S.

$$\begin{array}{lllcllll}
 & & s^5 & : & 9 & 10 & -9 & \text{.... Row-1} \\
-\text{ to }+ & +\text{ to }- & s^4 & : & -20 & -1 & -10 & \text{.... Row-2} \\
 & & s^3 & : & 9.55 & -13.5 & & \text{.... Row-3} \\
 & +\text{ to }- & s^2 & : & -29.3 & -10 & & \text{.... Row-4} \\
 & & s^1 & : & -16.8 & & & \text{.... Row-5} \\
 & & s^0 & : & -10 & & & \text{.... Row-6} \\
 & & & & \text{Column-1} & & &
\end{array}$$

$$s^3: \frac{-20\times 10-(-1)\times 9}{-20} \quad \frac{-20\times(-9)-(-10)\times 9}{-20}$$

$$s^3: 9.55 \quad -13.5$$

$$s^2: \frac{9.55\times(-1)-(-13.5)\times(-20)}{9.55} \quad \frac{9.55\times(-10)}{9.55}$$

$$s^2: -29.3 \quad -10$$

$s^1: \dfrac{-29.3\times(-13.5)-(-10)\times 9.55}{-29.3}$ $s^1: \ -16.8$
$s^0: \dfrac{-16.8\times(-10)}{-16.8}$ $s^0: \ -10$

By examining the elements of first column of Routh array it is observed that there are three sign changes and so three roots are lying on the right half of s-plane and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is unstable.
2. Three roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 5.5

Determine the range of K for stability of unity feedback system whose open loop transfer function is $G(s) = \dfrac{K}{s\,(s+1)\,(s+2)}$.

SOLUTION

The closed loop transfer function $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$= \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}}$$

$$= \frac{K}{s(s+1)(s+2)+K}$$

The characteristic equation is,

$$s(s+1)(s+2)+K=0$$

$$s(s^2+3s+2)+K=0$$

$$s^3+3s^2+2s+K=0$$

The Routh array is constructed as shown below.

The highest power of s in the character1stic polynomial is odd number, hence form the first row using the co-efficient of odd powers of s and form the second row using the coefficients of even powers of s.

$$\begin{array}{lcll} s^3 & : & \boxed{\begin{array}{c} 1 \\ 3 \\ \frac{6-K}{3} \\ K \end{array}} & \begin{array}{c} 2 \\ K \\ \\ \\ \end{array} \\ s^2 & : & & \\ s^1 & : & & \\ s^0 & : & \text{Column-1} & \end{array}$$

$$\begin{array}{|l|} \hline s^1 : \dfrac{3\times 2 - K\times 1}{3} \\ s^1 : \dfrac{6-K}{3} \\ \hline s^0 : \dfrac{\frac{6-K}{3}\times K - 0\times 3}{(6-K)/3} \\ s^0 : K \\ \hline \end{array}$$

For the system to be stable there should not be any sign Change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, K> 0

From s^1 row, for the system to be stable, $\frac{6-K}{3}>0$

For $\frac{6-K}{3}>0$, the value of K should be less than 6.

The range of K for the system to be stable is O<K<6.

RESULT

The value of K is in the range 0<K<6 for the system to be stable.

5.4 ROOT LOCUS

The root locus technique was introduced by W.R. Evans in 1948 for the analysis of control systems. **The root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters.**

Consider the open loop transfer function of system $G(S) = \frac{K}{s(s+p_1)(s+p_2)}$

The closed loop transfer function of the system with unity feedback is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+p_1)(s+p_2)}}{1+\frac{K}{s(s+p_1)(s+p_2)}}$$

$$= \frac{K}{s(s+p_1)(s+p_2)+K}$$

The denominator polynomial of C(s)/R(s) is the characteristic polynomial of the system. The characteristic equation is given by,

$$s(s + p_1)(s + p_2) + K = 0$$

The roots of characteristic equation is a function of open loop gain K. [In other words, the roots of characteristic equation depend on open loop gain K]. When the gain K is varied from 0 to ∞, the roots of characteristic equation will take different values.

When K = 0, the roots are given by open loop poles.

when K→ ∞ the roots will take the value of open loop zeros.

The path taken by the roots of characteristic equation when open loop gain K is varied from 0 to ∞ are called root loci (or the path taken by a root of characteristic equation when open loop gain K is varied from 0 to ∞ is called root locus).

Note: In general the roots of characteristic equation can be varied by varying any other system parameter other than gain.

5.4.1 CONSTRUCTION OF ROOT LOCUS:

The exact root locus is sketched by trial and error procedure. In this method, the poles and zeros of G(s)H(s) are located on the s-plane on a graph sheet and a trial point $s = s_a$ is selected. Determine the angles of vectors drawn from poles and zeros to the trial point. From the angle criterion, determine the angle to be contributed by these vectors to make the trial point as a point on root locus. Shift the trial point suitably so that the angle criterion is satisfied.

Several points are determined using the above procedure. Join the points by a smooth curve which is the root locus. The value of K for a root can be obtained from the-magnitude criterion.

The trial arid error procedure for sketching root locus is tedious. A set of rules have been developed to reduce the task involved in sketching root locus and to develop a quick approximate sketch. From the approximate sketch, a more accurate root locus can be obtained by a few trials.

5.4.2 RULES FOR CONSTRUCTION OF ROOT LOCUS

Rule1: The root locus is symmetrical about the real axis.

Rule 2: Each branch of the root locus originates from an open-loop pole corresponding to K = 0 and terminates at either on a finite open loop zero (or open loop zero at infinity) corresponding to K = ∞. The number of branches of the root locus terminating on infinity is equal to n-m, (i.e. The number of open loop poles minus the number of finite zeros)

Rule 3: Segments of the real axis having an odd number of real axis open-loop poles plus zeros to their right are parts of the root locus.

Rule 4: The n - m root locus branches that tend to infinity, do so along straight line asymptotes making' angles with the real axis given by,

$$\phi = \frac{\pm 180°[2q+1]}{(n-m)}$$

q=0, 1, 2...., (n-m).

Rule 5: The point of intersection of the asymptotes with the real axis is at s = σA where,

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Rule 6: The breakaway and break in points of the root locus are determined from the roots of the equation dK/ds =0. If r numbers of branches of root locus meet at a point, then they break away at an angle of ±180°/r.

Rule 7: The angle of departure from a complex open-loop pole is given by

$$\phi_P = \pm 180°(2q+1) + \phi;\ q=0,1,2, \ldots\ n-m$$

where ϕ is the net angle contribution at the pole by all other open loop poles and zeros. Similarly, the angle of arrival at a complex open loop zero is given by,

$$\phi_Z = \pm 180°(2q+1) + \phi;\ q=0,1,2, \ldots\ n-m$$

Where ϕ is the net angle contribution at the zero by all other open-loop poles and zeros.

Rule 8: The points of intersection of root locus branches with the imaginary axis can be determined by use of the Routh criterion. Alternatively, they can be evaluated by letting s = jω in the characteristic equation and equating the real part and imaginary part to zero, to solve for ω and K. The values of ω are the intersection points on imaginary axis and K is the value of gain at the intersection points.

Rule 9: The open-loop gain K at any point s = s. on the root locus is given by,

$$K = \frac{\prod_{i=1}^{n} |s_a + p_i|}{\prod_{i=1}^{m} |s_a + z_i|}$$

$$= \frac{\text{Product of vector lengths from open loop poles to the point } s_a}{\text{Product of vector lengths from open loop zeros to the point } s_a}$$

Note: The length of vector should be measured to scale. If there is no finite zero then the product of vector lengths from zeros is equal to 1.

5.4.3 TYPICAL SKETCHES OF ROOT LOCUS:

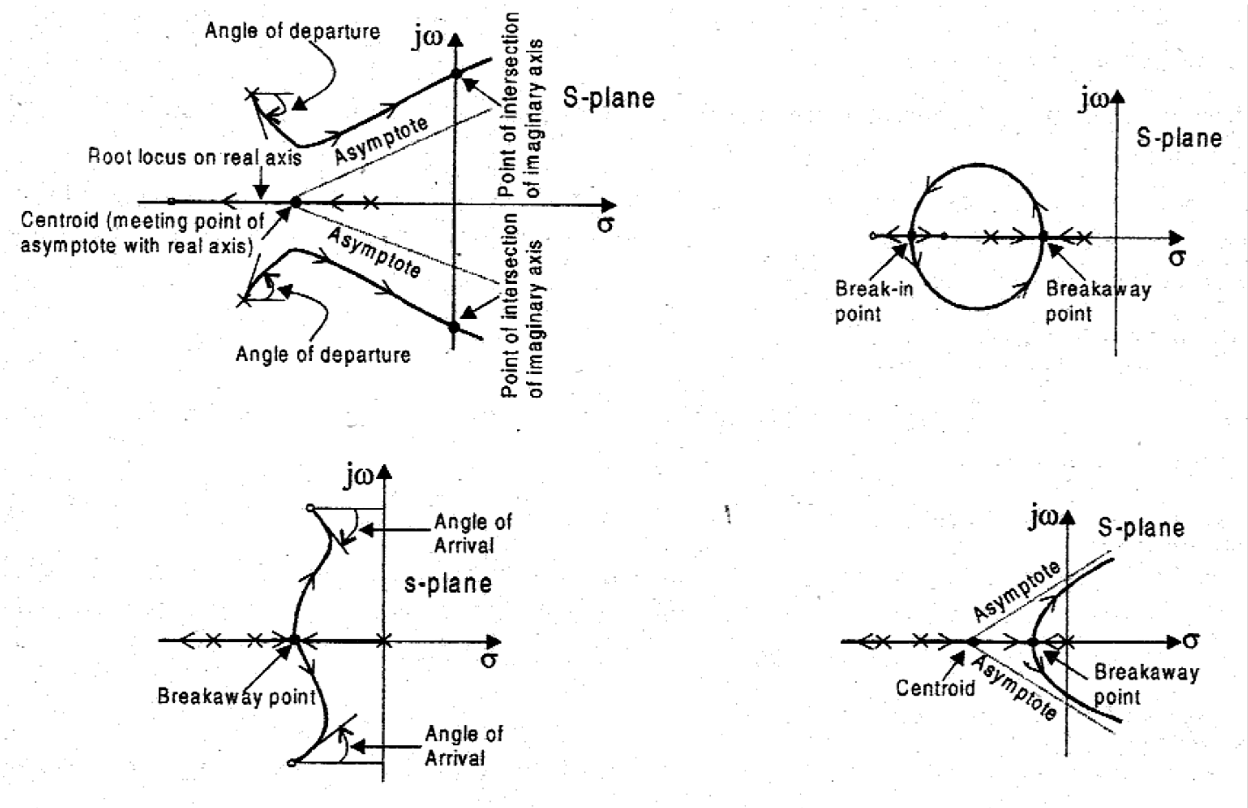


Fig 5.3

5.4.4 EXPLANATION FOR VARIOUS STEPS IN THE PROCEDURE FOR CONSTRUCTING ROOTS LOCUS:

Step 1: Location of poles and zero:

Draw the real and imaginary axis on an ordinary graph sheet and choose same scales both on real and imaginary axis.

The poles are marked by cross "X" and zeros are marked by small circle "0". The number of root locus branches is equal to number of poles of open loop transfer function. The origin of a root locus is at a pole and the end is at a zero.

Let, n = number of poles

m = number of zeros

Now m root locus branches end at finite zeros. The remaining n-m root locus branches will end at zeros at infinity,

Step 2: Root locus on real axis

To determine the part of root locus on real axis, take a test point on real axis. If the total number of poles and zeros on the real axis to the right of this test point is odd number, then the test point lies on the root locus. If it is even then the test point does not lie on the root locus.

Step 3: Angles of asymptotes and centroid

If 'n' is the number of poles and m is number of zeros, then n-m root locus branches will terminate at zeros at infinity.

These n-m root locus branches will go along an asymptotic path and meets the asymptotes at infinity. Hence number of asymptotes is equal to number of root locus branches going to infinity. The angles of asymptotes and the centroid are given by the following formula.

$$\text{Angles of asymptotes} = \frac{\pm 180\,(2q+1)}{n-m}$$

Where, q = 0, 1, 2, 3, ... (n-m)

$$\text{Centroid (meeting point of asymptote with real axis)} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Step 4: Breakaway and Break in points

The breakaway or break in points either lie on real axis or exist as complex conjugate pairs. If there is a root locus on real axis between 2 poles then there exists a breakaway point. If there is a root locus on real axis between 2 zeros then there exist a break in point, if there is a root locus on real axis between pole and zero, then there may be or may not be breakaway or break in point.

Let the characteristic equation be in the form,

$$B(s) + K\,A(s) = 0$$

$$\therefore K = \frac{-B(s)}{A(s)}$$

The breakaway and break in point is given by roots of the equation dK/ds = 0. The roots of dK/ds = 0 are actual breakaway or break in point provided for this value of root, the gain K should be positive and real.

Step 5: Angle of departure and angle of arrival

Angle of Departure (from a complex pole A) = 180°- (sum of angles of vector to the complex pole A from other poles) + sum of angles of vectors to the complex pole A from zeros)

If poles are complex then they exist only as conjugate pairs. Consider the two complex conjugate poles A and A*,

Angle of departure at pole A = 180°- ($\theta 1 + \theta 3 + \theta 5$) + ($\theta 2 + \theta 4$)

Angle of departure at pole A* = - [Angle of departure at pole A]

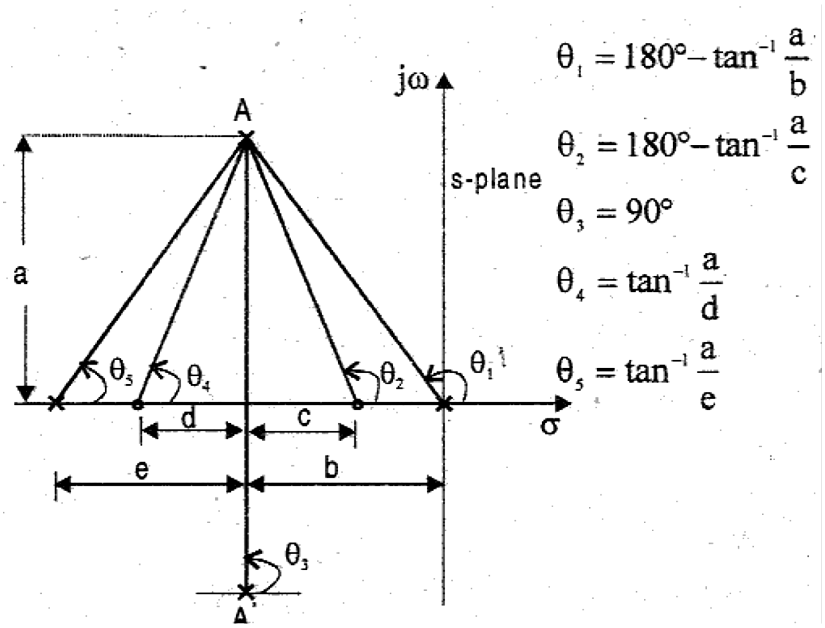


Fig 5.4

Angle of arrival (from a complex pole A) = 180°- (sum of angles of vector to the complex pole A from other zeros) + (sum of angles of vectors to the complex pole A from poles)

If zeros are complex then they exist only as conjugate pairs. Consider the two complex conjugate zeros A and A*,

Angle of arrival at pole A = 180°-($\theta 1 + \theta 3$) + ($\theta 2 + \theta 4 + \theta 5$)

Angle of arrival at pole A* = - [Angle of departure at zero A]

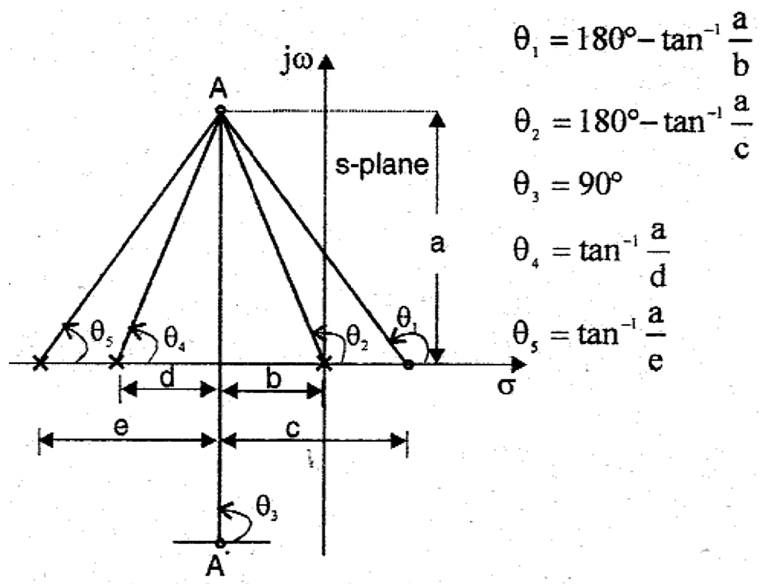


Fig 5.5

Step 6: Point of intersection of root locus with imaginary axis

The point where the root loci intersects the imaginary axis can be found by following three methods.

1. By Routh Hurwitz array.

2. By trial and error approach.

3. Letting s=jω the characteristic equation and separate the real part and imaginary part. Two equations are obtained one by equating real part to zero and the other by equating imaginary' part to zero. Solve the two equations for ω and K: The value of ω gives the points where the root locus crosses imaginary axis. The value of K gives the value of gain K at the crossing points. Also, this value of K is the limiting value of K for stability of the system.

Step 7: Test points and root locus

Choose a test point using a protractor roughly estimate the angles of vectors drawn to this point and adjust the point to satisfy angle criterion. Repeat the procedure for few more test points. Sketch the root locus from the knowledge of typical sketches and the information obtained in steps 1 through 6.

Note: In practice the approximate root locus can be sketched from the information steps 1 through 6 and from the knowledge of typical sketches of root locus.

5.4.5 DETERMINATION OF OPEN LOOP GAIN FOR A SPECIFIED DAMPING OF THE DOMINANT ROOTS:

The dominant pole is a pair of complex conjugate pole which decides the transient response of the system. In higher order systems, the dominant poles are given by the poles which are very close to origin provided all other poles are lying far away from the dominant poles. The poles which are far away from the origin will have less effect on transient response of the system.

The transfer function of higher order systems can be approximated to a second order transfer function. The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

The dominant poles are given by the roots of quadratic factor $s^2 + 2\delta\omega ns + \omega n^2 = 0.$

$$s_d = -2\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2}$$

$$\alpha = \cos^{-1}\delta$$

To fix a dominant pole on root locus, draw a line at an angle of $\cos^{-1}\delta$ with respect to negative real axis. The meeting point of this line with root locus will give the location of dominant pole. The value of K corresponding to dominant pole can be obtained from magnitude condition.

The gain K corresponding to dominant pole,

$$s_d = \frac{\text{product of lendth of vectors from open loop poles to dominant pole}}{\text{product of lendth of vectors from open loop zeros to dominant pole}}$$

EXAMPLE 5.6

A unity feedback control system has an open loop transfer function $G(s) = \frac{K}{s(s^2+4s+13)}$

Sketch the root locus.

SOLUTION

Step 1: To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s^2 + 4s + 13) = 0.$

The roots of the quadratic are, $s = \frac{-4\pm\sqrt{4^2-4\times3}}{2}$ = $-2\pm j3$

The poles are 0, -2+j3 and -2-j3.

The poles are marked by X.

Step 2: To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as a bold line in fig

Step 3: To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches is three. There is no finite zero. Hence all the three root locus branches end at zeros at infinity. The number of asymptotes required is three.

Angles of asymptotes = $\frac{\pm 180°(2q+1)}{n-m}$ hence n=3 and m=0.

If q=0, Angles = $\pm\frac{180°}{3} = \pm 60°$

If q=1, Angles = $\pm\frac{180°\times 3}{3} = \pm 180°$

If q=2, Angles = $\pm\frac{180°\times 5}{3} = \pm 60°$

If q=1, Angles = $\pm\frac{180°\times 7}{3} = \pm 60°$

Centroid = $\frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$

$= \frac{0-2+j3-2-j3-0}{3}$ = -4/3 = -1.33

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig.

Step 4: To find the break away and break in points

The closed loop transfer function of the system with unity feedback is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s^2+4s+13)}}{1+\frac{K}{s(s^2+4s+13)}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+4s+13)+K}$$

The characteristic equation is $s(s^2+4s+13)+K = 0$

$$s^3+4s^2+13s+K = 0$$

$$K = -s^3-4s^2-13s$$

$$dK/ds = -(3s^2+8s+13)$$

Put dK/ds = 0; $-(3s^2+8s+13) = 0$

$$s = \frac{-8\pm\sqrt{64-4\times 13\times 3}}{2\times 3} \qquad = -1.33\pm j1.6$$

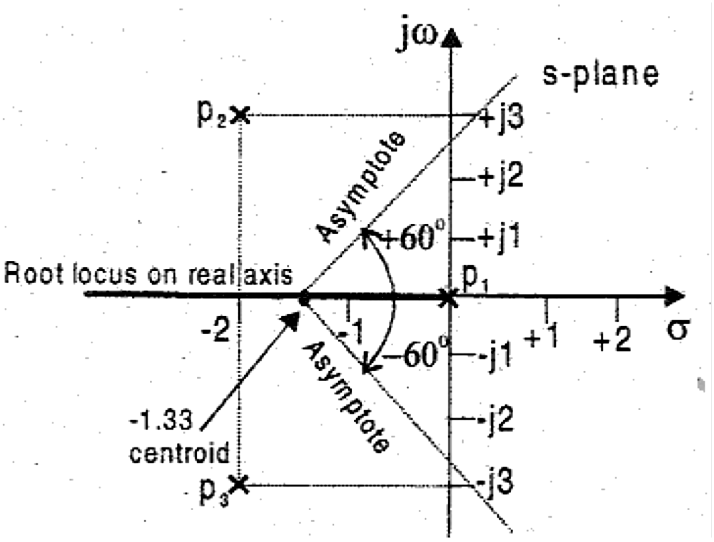


Fig 5.6

Check for K,

When s= -1.33+j1.6, $K = [(-1.33+j1.6)^3+4(-1.33+j1.6)^2+13(-1.33+j1.6)]$

$\neq$ Positive and real

When s= -1.33-j1.6, $K = [(-1.33-j1.6)^3+4(-1.33-j1.6)^2+13(-1.33-j1.6)]$

$\neq$ Positive and real

Hence the values are not real and positive; the points are not actual breakaway or break in points. The root locus has neither breakaway nor break in point.

Step 5: To find angle of departure

Let us consider the complex pole A, draw vectors from all other poles to the pole A. let the angles of these vectors be θ1 and θ2.

Here, $\theta 1 = 180° - \tan^{-1}(3/2) = 123.7°$

$\theta 2 = 90°$

Angle of departure from the complex pole A = $180° - (\theta 1 + \theta 2)$

$= 180° - (123.7° + 90°) = -33.7°$

Angle of departure at complex pole A* = +33.7°

Mark the angles of departure at complex poles using protractor.

Step 6: To find the crossing point on imaginary axis

The characteristic equation is given by

$s^3+4s^2+13s+K = 0$

Put s= jω, $(j\omega)^3+4(j\omega)^2+13(j\omega) +K = 0$

$-j\omega^3-4j\omega^2+13j\omega+K = 0$

On equating imaginary part to zero, $-\omega^3+13\omega = 0$

$$\omega^2 = 13, \qquad \omega = \pm 3.6 \text{ rad/sec}$$

On equating real parts to zero,

$$-4\omega^2 + K = 0$$

$$K = 4\omega, \; K = 4(13) = 52$$

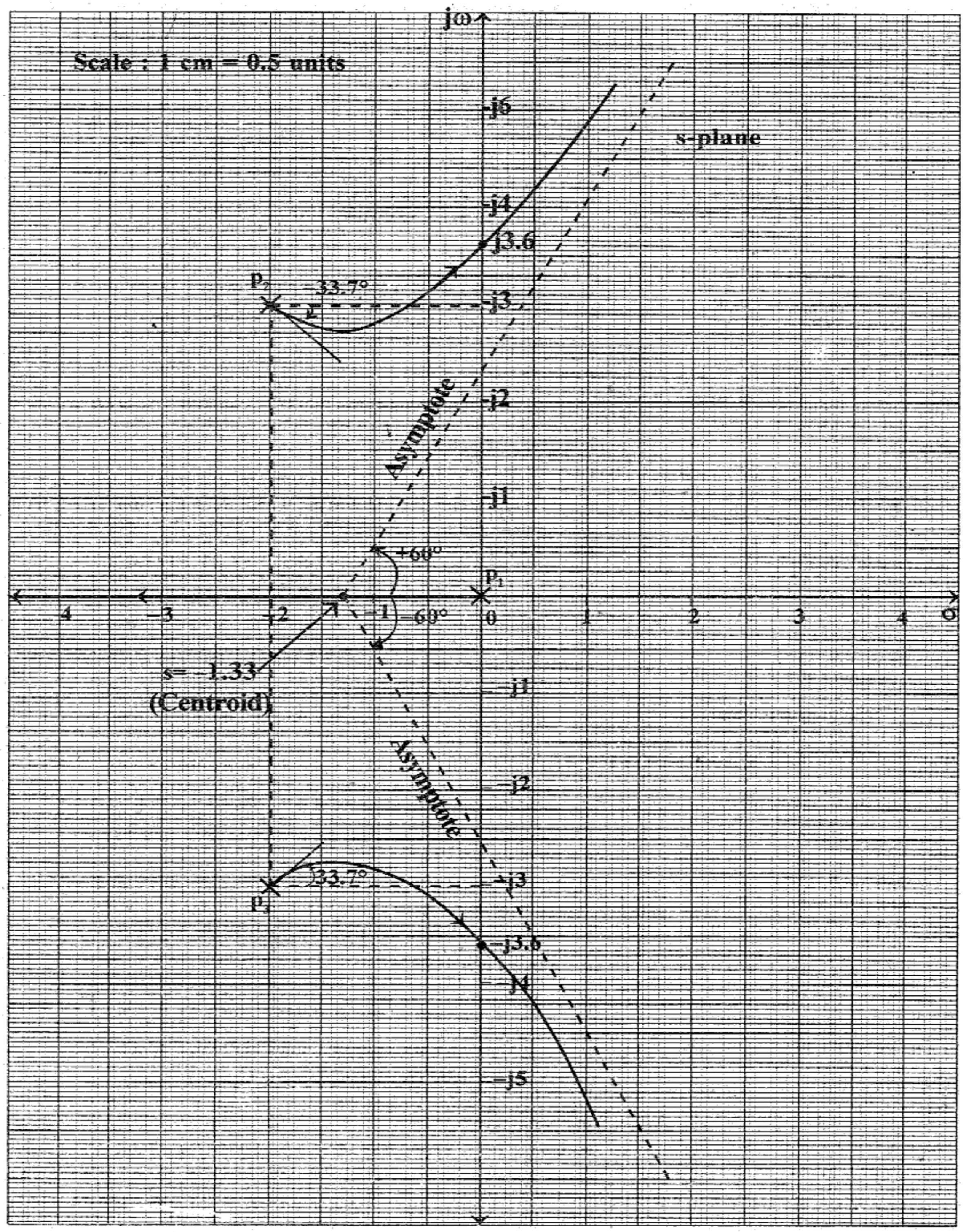


Fig 5.7

The crossing point of root locus is ± j3.6. The value of K at this crossing point is K = 52. (This is the limiting value of K for the stability of the system)

The complete root locus sketch is shown in figure 5.7. The root locus has three branches. One branch starts at the pole at origin and travel through negative real axis to meet the zero at infinity. The other two root locus branches start at complex poles (along the angle of departure) cross the imaginary axis at ± j3.6 and travel parallel to asymptotes to meet the zeros at infinity.

EXAMPLE 5.6

A unity feedback control system has an open loop transfer function G(s) = $\frac{K}{s(s+2)(s+4)}$ Sketch the root locus. Find the value of K so that the damping ratio of the closed loop system is 0.5.

SOLUTION

Step1: To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s+2)(s+4) = 0.$

The poles are s= 0, -2 and -4.

The poles are marked by X.

Step 2: To find the root locus on real axis

There are three poles on real axis. Choose a test point on real axis between s=0 and a=-2. To the right of this point the total number of real poles and zeros is one which is an odd number. Hence the real axis between s=0 and s= -2 will be a part of root locus.

Choose a test point on real axis between s = -2 and s = -4. To the right of this point, the total number of real poles and zeros is two which is an even number. Hence the real axis between s= -2 and s = - 4 will not be a part of root locus.

Choose a test point on real axis to the left of s=-4. To the right of this point, the total number of real poles and zeros is three, which is an odd number. Hence the entire negative real axis from s =-4 to ∞ will be a part of root locus.

The root locus on real axis are shown as bold lines in figure 5.8

Step 3: To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches is three. There is no finite zero. Hence all the three root locus branches end at zeros at infinity. The number of asymptotes required is three.

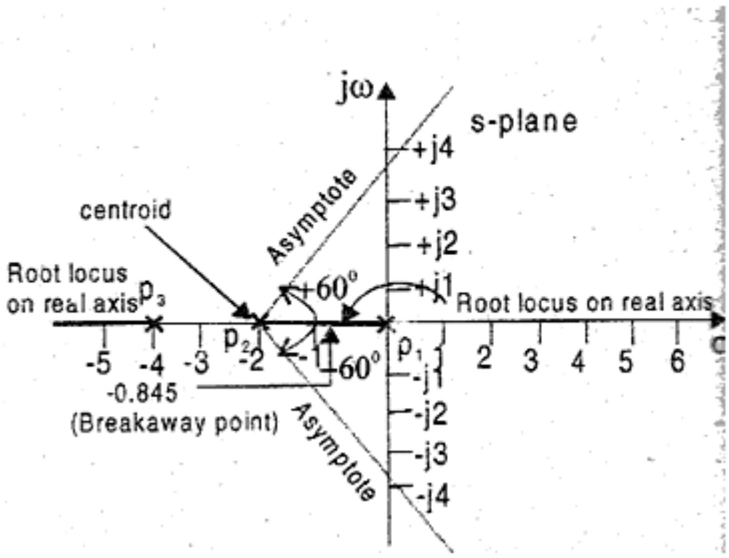


Fig 5.8

Angles of asymptotes = $\frac{\pm 180°(2q+1)}{n-m}$ hence n=3 and m=0.

If q=0, Angles = $\pm\frac{180°}{3} = \pm 60°$

If q=1, Angles = $\pm\frac{180°\times 3}{3} = \pm 180°$

If q=2, Angles = $\pm\frac{180°\times 5}{3} = \pm 60°$

If q=1, Angles = $\pm\frac{180°\times 7}{3} = \pm 60°$

Centroid = $\frac{\text{sum of poles} - \text{sum of zeros}}{n-m} = \frac{0-2-4-0}{3}$ = -2

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in figure 5.8.

Step 4: To find the break away and break in points

The closed loop transfer function of the system with unity feedback is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}}$$

$$= \frac{K}{s(s+2)(s+4)+K}$$

The characteristic equation is $s(s+2)(s+4)+K = 0$

$$s^3+6s^2+8s+K = 0$$

$$K = -s^3-6s^2-8s$$

$$dK/ds = -(3s^2+12s+8)$$

$$\text{put } dK/ds = 0;\ -(3s^2+12s+8) = 0$$

$$s = \frac{-12 \pm \sqrt{144-4\times3\times8}}{2\times3}$$

$= -0.845$ or -3.154

Check for K,

When s= -0.845, $K = [(-0.845)^3+6(-0.845)^2+8(-0.845)]$

= positive and real for K = -0.845, this point is actual breakaway point.

When s= -3.154, $K = [(-3.154)^3+6(-3.154)^2+8(-3.154)]$

= -3.08

$\neq$ Positive, hence K=-3.154 is not a breakaway point.

Step 5: To find angle of departure

Since there is no complex pole or zero, we need not find angle of departure or arrival.

Step 6: To find the crossing point on imaginary axis

The characteristic equation is given by

$$s^3+6s^2+8s+K = 0$$

Put s= jω, $(j\omega)^3+6(j\omega)^2+8(j\omega)+K = 0$

$$-j\omega^3-6j\omega^2+8j\omega+K = 0$$

On equating imaginary part to zero,

$-\omega^3+8\omega = 0$

$\omega^2 = 8$

$\omega = \pm 2.8$

On equating real parts to zero,

$-6\omega^2+K = 0$

$K = 6\omega^2$

$K = 6(8) = 48$

The crossing point of root locus is ± j2.8. The value of K at this crossing point is K = 48. (This is the limiting value of K for the stability of the system),

The complete root locus sketch is shown in fig. The root locus has three branches. One branch starts at the pole at s = - 4 and travel through negative real axis to meet the zero at Infinity. The other two root locus branches start at s = 0 and s = - 2 and travel through negative real axis, breakaway from real axis at s= -0.845, then crosses imaginary axis at s= ± j2.8 and travel parallel to asymptotes to meet the zeros at infinity.

To find the value of K corresponding to δ = 0.5

Given that δ= 05

Let $a = \cos^{-1}\delta = \cos^{-1} 0.5 = 60°$

Draw a line OP, such that the angle between line OP and negative real axis is 60° (a= 60°) as shown in fig 4.23.2. The meeting point of the line OP and root locus gives the dominant pole, s_d.

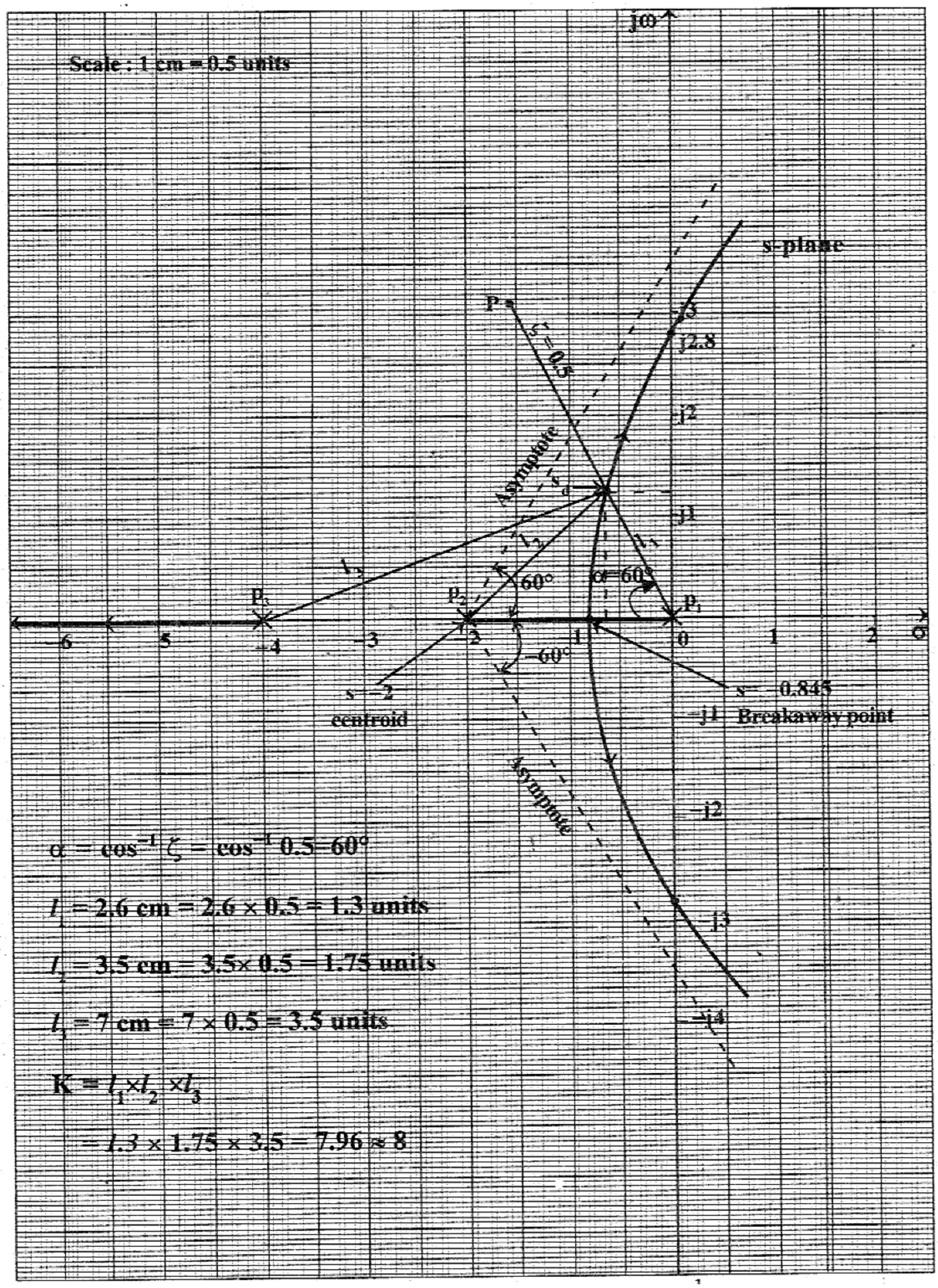


Fig 5.9

REVIEW QUESTIONS

PART A

1. Define stability.
2. What is stable system?
3. What is absolutely stable system?
4. What is limitedly stable system?
5. Define asymptotically stable system.
6. Define BIBO stability.
7. What is root locus?
8. What is root locus on real axis?
9. Write the formula for find centroid.
10. Write the formula for angle of asymptote.

PART B

1. Comment the location of roots on s-plane for stability.
2. State Routh Stability criterion.
3. Determine the stability of the system $S^6+S^5-2S^4-3S^3-7S^2-4S-4=0$.Comment on the location of the roots of characteristic equation.
4. Draw the typical sketches of Root locus.
5. What is break away and break in points and how can you find it?

PART C

1. Construct Routh array and determine the stability of the, system represented by the characteristic equation, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.Comment on the location of the roots of characteristic equation.
2. Construct Routh array and determine the stability of the, system represented by the characteristic equation, $s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$. Comment on the location of the roots of characteristic equation.
3. Construct Routh array and determine the stability of the, system represented by the characteristic equation, $s^6+s^5+3s^4+3s^3+3s^2+2s+1=0$.Comment on the location of the roots of characteristic equation.
4. Determine the range of K for stability of unity feedback system whose open loop transfer function is $G(s) = \frac{K}{s(s+1)\,(s^2+s+1)}$

5. Explain the step by step procedure for constructing Root locus.
6. A unity feedback control system has an open loop transfer function $G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$

   Sketch the root locus.
7. A unity feedback control system has an open loop transfer function $G(s) = \frac{K(s+4)}{s(s+0.5)(s+2)}$ Sketch the root locus.