GOVERNMENT OF TAMILNADU
DIRECTORATE OF TECHNICAL EDUCATION
CHENNAI – 600 025

STATE PROJECT COORDINATION UNIT

Diploma in Civil Engineering

Course Code: 31041
M – Scheme

e-TEXTBOOK

on

THEORY OF STRUCTURES

for

IV Semester DCE

Convener for CIVIL Discipline:

Mr.N. Muralikrishniah, M.E,
Principal / IPT &
Principal (Addl. Charge)
Central Polytechnic College,
Taramani, Chennai – 600 113
Email : muralinapa@gmail.com

Team Members for Theory of Structures

1. **Mr.S.Loganathan, M.E.,**
   HoD /CIVIL,
   Thiagarajar Polytechnic College,
   Salem – 636 005
2. **Mr.V.Amalraj, M.E.,**
   Lecturer (Sel.Gr.) / CIVIL,
   Government Polytechnic College for Women,
   Coimbatore – 641 044
3. **Mr.T.Thangamani, M.E.,**
   Lecturer (Sel.Gr.) / CIVIL,
   Srinivasa Subbaraya Polytechnic College,
   Puttur, Sirkali – 609 108

Validated by

Dr.V.Karthikeyan, M.E., Ph.D., MCMI(UK)., FIE.,
Principal,
Thiagarajar Polytechnic College,
Salem – 636 005.

DETAILED SYLLABUS

UNIT-I

1.1 SLOPE AND DEFLECTION OF BEAMS

Deflected shapes / Elastic curves of beams with different support conditions –Definition of Slope and Deflection- Flexural rigidity and Stiffness of beams- Mohr's Theorems – Area Moment method for slope and deflection of beams – Derivation of expressions for maximum slope and maximum deflection of standard cases by area moment method for cantilever and simply supported beams subjected to symmetrical UDL & point loads – Numerical problems on determination of slopes and deflections at salient points of Cantilevers and Simply supported beams from first principles and by using formulae

1.2 PROPPED CANTILEVERS

Statically determinate and indeterminate Structures- Stable and Unstable Structures- Examples- Degree of Indeterminacy-Concept of Analysis of Indeterminate beams - Definition of Prop –Types of Props- Prop reaction from deflection consideration – Drawing SF and BM diagrams by area moment method for UDL throughout the span, central and non-central concentrated loads – Propped cantilever with overhang – Point of Contra flexure.

UNIT-II

2.1 FIXED BEAMS – AREA MOMENT METHOD

Introduction to fixed beam - Advantages –Degree of indeterminacy of fixed beam- Sagging and Hogging bending moments – Determination of fixing end(support) moments(FEM) by Area Moment method – Derivation of Expressions for Standard cases – Fixed beams subjected to symmetrical and unsymmetrical concentrated loads and UDL – Drawing SF and BM diagrams for Fixed beams with supports at the same level (sinking of supports or supports at different levels are not included) – Points of Contra flexure – Problems- Determination of Slope and Deflection of fixed beams subjected to only symmetrical loads by area moment method – Problems.

2.2 CONTINUOUS BEAMS – THEOREM OF THREE MOMENTS METHOD

Introduction to continuous beams – Degree of indeterminacy of continuous beams with respect to number of spans and types of supports –Simple/Partially fixed / Fixed supports of beams- General methods of analysis of Indeterminate structures – Clapeyron's theorem of three moments – Application of Clapeyron's theorem of three moments for the following cases – Two span beams with both ends simply supported or fixed – Two span beams with one end fixed and the other end simply supported – Two span beams with one end simply supported or fixed and other end overhanging –Determination of Reactions at Supports- Application of Three moment equations to Three span Continuous Beams and Propped cantilevers –Problems- Sketching of SFD and BMD for all the above cases.

UNIT-III

3.1 CONTINUOUS BEAMS – MOMENT DISTRIBUTION METHOD

Introduction to Carry over factor, Stiffness factor and Distribution factor –Stiffness Ratio or Relative Stiffness- Concept of distribution of un balanced moments at joints - Sign conventions – Application of M-D method to Continuous beams of two / three spans and to Propped cantilever (Maximum of three cycles of distribution sufficient) –Finding Support Reactions- Problems - Sketching SFD and BMD for two / three span beams.

3.2 PORTAL FRAMES – MOMENT DISTRIBUTION METHOD

Definition of Frames – Types – Bays and Storey - Sketches of Single/Multi Storey Frames, Single/Multi Bay Frames- Portal Frame – Sway and Non- sway Frames- Analysis of Non sway (Symmetrical) Portal Frames for Joint moments by Moment Distribution Method and drawing BMD only– Deflected shapes of Portal frames under different loading / support conditions.

UNIT-IV

4.1 COLUMNS AND STRUTS

Columns and Struts – Definition – Short and Long columns – End conditions – Equivalent length / Effective length– Slenderness ratio – Axially loaded short column - Axially loaded long column – Euler’s theory of long columns – Derivation of expression for Critical load of Columns with hinged ends – Expressions for other standard cases of end conditions (separate derivations not required) – Problems – Derivation of Rankine’s formula for Crippling load of Columns– Factor of Safety- Safe load on Columns- Simple problems.

4.2 COMBINED BENDING AND DIRECT STRESSES

Direct and Indirect stresses – Combination of stresses – Eccentric loads on Columns – Effects of Eccentric loads / Moments on Short columns – Combined direct and bending stresses – Maximum and Minimum stresses in Sections– Problems – Conditions for no tension – Limit of eccentricity – Middle third rule – Core or Kern for square, rectangular and circular sections – Chimneys subjected to uniform wind pressure –Combined stresses in Chimneys due to Self weight and Wind load- Chimneys of Hollow square and Hollow circular cross sections only – Problem

UNIT-V

5.1 MASONRY DAMS

Gravity Dams – Derivation of Expression for maximum and minimum stresses at Base – Stress distribution diagrams – Problems – Factors affecting Stability of masonry dams – Factor of safety- Problems on Stability of Dams– Minimum base width and maximum height of dam for no tension at base – Elementary profile of a dam – Minimum base width of elementary profile for no tension.

5.2 EARTH PRESSURE AND RETAINING WALLS

Definition – Angle of repose /Angle of Internal friction of soil– State of equilibrium of soil – Active and Passive earth pressures – Rankine’s theory of earth pressure – Assumptions – Lateral earth pressure with level back fill / level surcharge (Angular Surcharge not required)– Earth pressure due to Submerged soils – (Soil retained on vertical back of wall only) – Maximum and minimum stresses at base of Trapezoidal Gravity walls – Stress distribution diagrams – Problems – Stability of earth retaining walls – Problems to check the stability of walls- Minimum base width for no tension.

UNIT – I

1.1 SLOPE AND DEFLECTION OF BEAMS

1.1 SLOPE AND DEFLECTION OF BEAMS

Deflected shapes / Elastic curves of beams with different support conditions –Definition of Slope and Deflection- Flexural rigidity and Stiffness of beams- Mohr's Theorems – Area Moment method for slope and deflection of beams – Derivation of expressions for maximum slope and maximum deflection of standard cases by area moment method for cantilever and simply supported beams subjected to symmetrical UDL & point loads – Numerical problems on determination of slopes and deflections at salient points of Cantilevers and Simply supported beams from first principles and by using formulae.

CHAPTER 1

1.1 INTRODUCTION

1. Beam

A structural member which is acted upon by a system of external loads at right angles to its axis is known as beam. Generally, a beam is a horizontal member to support floor slabs, secondary beams, walls, stairs etc.

2. Classification of structure

In general, the following are two types of structures

a) According to static equilibrium equation

i) Statically determinate structures

ii) Statically indeterminate structures

Further, the above structures are classified according to support conditions as presented below

b) According to support conditions

1. Cantilever beam
2. Simply supported beam
3. Propped cantilever beam
4. Overhanging beam
5. Fixed beam
6. Continuous beam

3. Shear force (S.F)

The Shear Force at any section of a beam is the algebric sum of all the forces acting either left or right of that section. It is denoted by F (or) SF. The symbol of SF is F(or)V(or)SF.

4. Bending moment (B.M)

The bending moment at any section of a beam is the algebric sum of all the moments of the forces acting either left (or) right of that section. It is denoted by B.M(or) M.

1.1.1 Deflected shapes of beam / Elastic line (or) elastic curve of beam

When a beam is subjected to transverse loads it develops shear force and bending moment at every cross section. Due to transverse load the beam gets deflected. The deflected configuration of the beam is known as deflected shape.

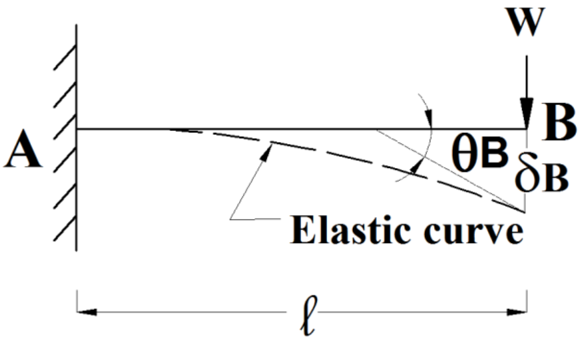


Fig 1.1 Elastic curve

Where, θB = Slope at B

SB = Deflection of free end(B)

(a) Elastic line (or) elastic curve of beam

The configuration of the longitudinal axis of the beam after bending takes place due to loading is called elastic curve. (or)

The edge view of the deflected neutral surface of a beam is known as elastic curve.

The deflected shape of various types of beam is presented below

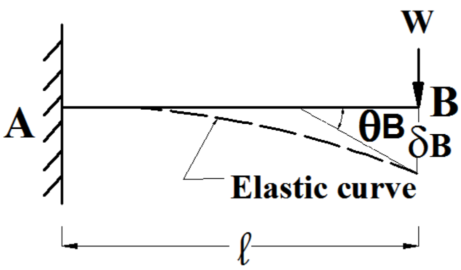


Fig 1.2 For a cantilever beam

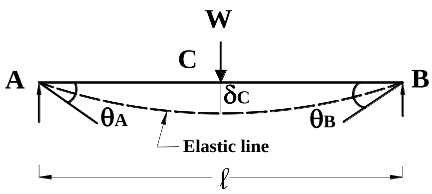


Fig 1.3 For a simply supported beam

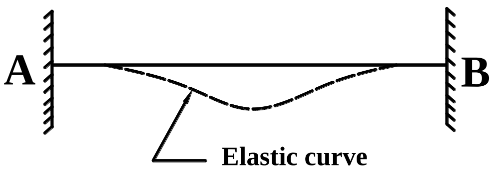


Fig 1.4 For a fixed beam

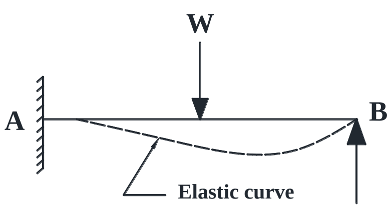


Fig 1.5 For a propped cantilever

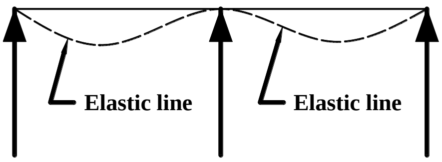


Fig 1.6 For a continuous beam

1.1.2 Slope and Deflection

(a) Slope (θ or i)

The angle made by the tangent at a point on the elastic curve with the horizontal is called the slope at the point. It is denoted by θ (or) i.

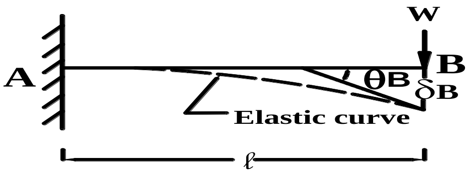


Fig 1.7

(b) Deflection (δ or y)

The vertical distance between the original axis to the elastic curve of the beam after loading is called deflection. It is denoted by δ (or) y.

1.1.3 Flexural rigidity and Stiffness of beams

(a) Flexural rigidity (EI)

The product of values of young's modulus and MI about Neutral axis is called flexural rigidity.

ie. Flexural rigidity = Young's modulus x M.I
= E x I = EI

The product of EI is expressed in $N.mm^2$ (or) $kN.m^2$

(b) Stiffness

Stiffness of a beam is the property of resistance against rotation and deflection. The moment required to produce unit rotation of slope is called stiffness of the beam. It is depends upon the end conditions, flexural rigidity and span of the beam.

1. Stiffness factor, k $= \frac{4EI}{\ell}$ ⟶ for fixed ends
2. Stiffness factor, k $= \frac{3EI}{\ell}$ ⟶ for simply supported ends

1.1.3 Book Work - I

a) Differential Equation

Derive the differential equation of flexure (or) relation between slope deflection and radius of curvature.

ie $$\boxed{M = EI\frac{d^2y}{dx^2}}$$

Solution

Consider a small portion of PQ of length 'ds' of a beam as shown in fig.1.4

R = Radius of curvature

dθ = Angle bounded by the arc 'ds' at centre

We know,

Arc length ds = dx = R. dθ

$$\frac{1}{R} = \frac{d\theta}{dx} \rightarrow (1)$$

$$\tan\theta = \frac{dy}{dx} \qquad \therefore \frac{dy}{dx} = \theta$$

Differentiate w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx} \rightarrow (2)$$

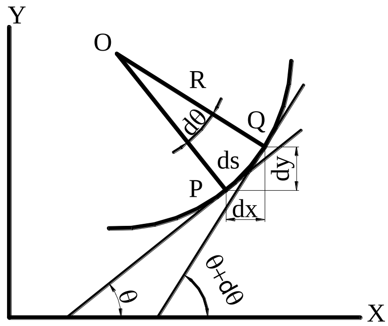


Fig 1.8

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R} \qquad \frac{M}{I} = \frac{E}{R}$$

$$\frac{1}{R} = \frac{M}{EI} \rightarrow (4)$$

$$\therefore \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$M = EI\,\frac{d^2y}{dx^2}$$

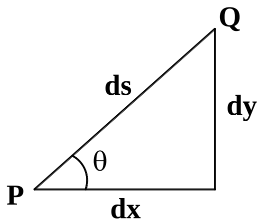


Fig 1.9

$$\boxed{M = EI\,\frac{d^2y}{dx^2}}$$ Differential equation of flexure.

Note:

$EI\,\frac{d^2y}{dx^2}$ = M → Flexural differential equation.

$\int EI\,\frac{d^2y}{dx^2}$ = slope = $EI\left(\frac{dy}{dx}\right)$

$\iint EI\,\frac{d^2y}{dx^2}$ = deflection = EI y

$\frac{d(M)}{dx} = \frac{d}{dx}\left(\frac{EI\,d^2y}{dx^2}\right)$ = F = shear force.

$\frac{d}{dx}\left(\frac{EI\,d^3y}{dx^3}\right)$ = w = Rate of loading

1.1.3 (b) Method of finding the slope and deflection

The following are the various methods for slope and deflection.

1. Mohr's area moment method
2. Double integration method

3. Macaulay's method
4. Strain energy method
5. Conjugate beam method

1.1.4 Area moment method

It is a simple method, Mohr's Theorem I & II are used for the determination of slope and deflection of beams at any section with reference to the B.M.D., hence it is called as Mohr's area moment method.

1.1.4 (a) Mohr's Theorem – I

It states that the change of slope between any two points on an elastic curve is equal to the area of BMD between the two points divided by flexural rigidity.

Slope = $\theta = \frac{A}{EI}$

(b) Mohr's Theorem – II

It states that the intercept taken on a vertical reference line of tangents at any two points on an elastic curve is equal to the moment area of BMD between these points about the reference line divided by flexural rigidity.

Y = $\frac{A\bar{x}}{EI}$ = deflection.

Where, A = Net area of BMD

$A\bar{x}$ = Net moment area of BMD.

EI = Flexural rigidity

<u>**Book Work – 2**</u>

State and prove Mohr's theorem for slope (Mohr's theorem – I)

1.1.4 (b) Mohr's Theorem - I

It states that the change of slope between any two points on an elastic curve is equal to the area of BMD between the two points divided by flexural rigidity.

$\theta = \frac{A}{EI}$ = slope

Where

A = Net area of BMD

EI = Flexural rigidity.

Proof

Consider a beam AB deflected as shown in fig.

Consider two points P & Q and draw tangents at P and Q.

PQ = dx = length of curve

$d\theta$ = change of slope

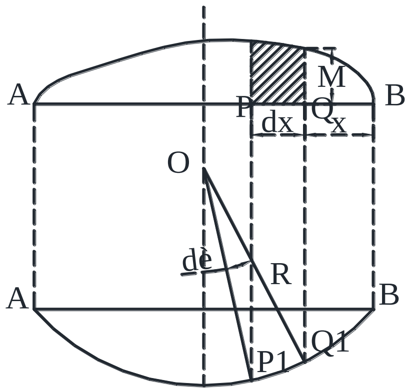


Fig 1.10

We know

dx = R. $d\theta$

$$\frac{1}{R} = \frac{d\theta}{dx} \quad \rightarrow (1)$$

But $$\frac{M}{I} = \frac{E}{R}$$

$$\frac{1}{R} = \frac{M}{EI} \quad \rightarrow (2)$$

$$\therefore \frac{M}{EI} = \frac{d\theta}{dx}$$

$$d\theta = \frac{M}{EI} \times dx$$

$$\therefore \theta_B = \frac{\Sigma M\, dx}{EI} = \frac{A}{I}$$

$$\boxed{\theta_B = \frac{A}{EI}}$$ Where ΣM. dx = A = Net area of BMD

<u>Book Work – 3</u>

State and prove Mohr's Theorem – II for deflection.

1.1.4 (c) Mohr's Theorem – II

It states that the intercept taken on a vertical reference line of tangents at any two points on an elastic curve is equal to the moment area of BMD between these points about the reference line divided by flexural rigidity.

ie $$\boxed{y = \frac{A\bar{x}}{EI}}$$

Where $A\bar{x}$ = Moment area of BMD

EI = Flexural rigidity

Proof

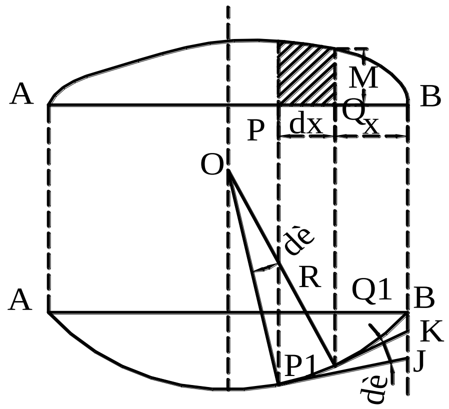
Consider a BMD as shown in fig.

$$d\theta = \frac{M\ dx}{EI}$$

$$d\theta\ \bar{x} = \frac{M\ dx}{EI}\ \bar{x} \qquad (A = \Sigma M\ dx)$$

$$\delta_{A/B} = \frac{\Sigma M\ dx}{EI}\ \bar{x} = \frac{A\bar{x}}{EI}$$

$$\boxed{\delta_{A/B} = \frac{A\bar{x}}{EI}}$$ deflection



<u>Fig 1.11</u>

Where M dx = area of BMD

$\bar{x}$ = centroid of BMD

A $\bar{x}$ = Moment area of BMD

1.1.5 Slope and Deflection for standard cases

Book Work – 4

1.1.5 (a) Cantilever beam with point load 'W' at free end.

Derive the max. slope and deflection for a cantilever beam with point load 'W' at free end.

Solution

Consider a cantilever beam loaded as shown in fig.
Draw BMD as shown in fig.
Draw tangents at A & B to the elastic curve.

By Mohr's Theorem - I

$$\theta = \frac{A}{EI}$$

A = Area of BMD

$$A = \frac{1}{2} \text{xbxh}$$

$$A = \frac{1}{2} \text{ x } \ell \text{ x } W\ell = \frac{W\ell^2}{2}$$

$$\theta_B = \frac{A}{EI} = \frac{1}{EI} \text{x} \left(\frac{W\ell^2}{2} \right)$$

$$\boxed{\theta_B = \frac{W\ell^2}{2EI}}$$

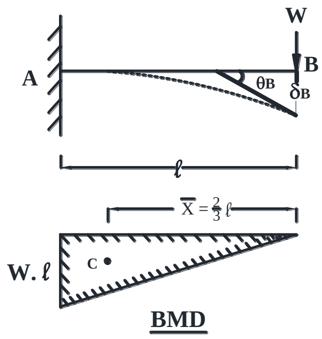


Fig 1.12

By Mohr's Theorem – II

$$\delta_B = \delta_{max} = \frac{A\bar{x}}{EI} = \frac{1}{EI} \left[\frac{W\ell^2}{2} \text{x} \frac{2}{3} \ell \right]$$

$\bar{x} =$ Centroid of BMDfrom B

$$\bar{x} = \frac{2}{3} \ell$$

$$\boxed{\delta_B = \delta_{max} = \frac{W\ell^3}{3EI}}$$ deflection at B

Book Work – 5

1.1.5 (a) Cantilever beam with udl (w/m)

Derive the max. slope and deflection for a cantilever beam with udl over entire span.

Solution

Consider a cantilever beam loaded as shown in fig.

Draw BMD

Let A = area of BMD between A & B

$$A = \frac{1}{3} \text{xbxh}$$

$$A = \frac{1}{3} \text{ x } \ell \text{ x } \frac{w\ell^2}{2} = \frac{w\ell^3}{6}$$

$\bar{x}$ = Centroid of BMD from B

$$\bar{x} = \frac{3}{4} \text{ x } \ell$$

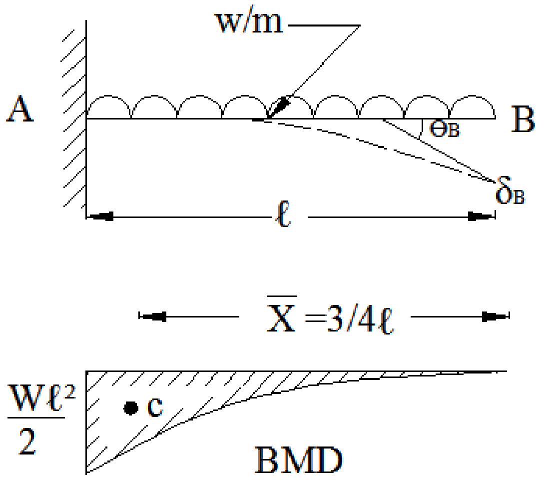


Fig 1.13

By Mohr's Theorem – I

$$\theta_B = \frac{A}{EI}$$

$$\theta_B = \frac{1}{EI}\left(\frac{w\ell^3}{6}\right) = \frac{w\ell^3}{6EI}$$

$$\boxed{\theta_B = \frac{w\ell^3}{6EI}}$$ slope at B

By Mohr's Theorem – II;

$$\delta_B = \delta_{max} = \frac{A\bar{x}}{EI}$$

$$\delta_B = \delta_{max} = \frac{1}{EI}\left(\frac{w\ell^3}{6} \text{x} \frac{3}{4}\ell\right)$$

Deflection at B = $\boxed{\delta_B = \delta_{max} = \left(\frac{w\ell^4}{8EI}\right)}$

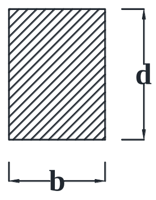
Solved Problems

Problem 1

A cantilever beam 2m span 200mm wide and 400mm deep. Carries a point load of 10 kN at free end. Find the max. slope and deflection by area moment method take $E = 2.0 \times 10^5$ N/mm^2.

Given data:

By area moment method



Span ℓ = 2m = 2000 mm

Wide b = 200mm

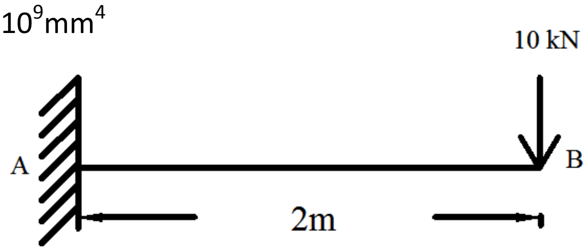
Depth d = 400mm

E = 2.0×10^5 N/mm^2

Solution

(i)Moment of inertia (I)

M.I. (I) $= \dfrac{bd^3}{12} = \dfrac{200\,x\,400^3}{12}$ = 1.067×10^9mm^4



(ii) Bending Moment

W. ℓ = 10 x 2 = 20 kN.m

Draw BMD as shown in fig.1.9

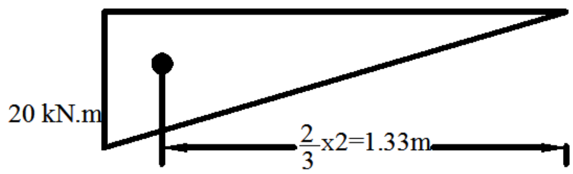
A = area of BMD

A $= \dfrac{1}{2}$ x b x h

A $= \dfrac{1}{2}$ x 2 x 20 = 20 kNm^2 = 20×10^9 Nmm^2

$\bar{x}$ = Centroid of BMD draw B

$\bar{x} = \dfrac{2}{3}$ x 2 = 1.33m=1.33 x 10^3 mm



BMD

(ii) Slope & Deflection

By Mohr's Theorem – I

Slope $\theta_B = \dfrac{A}{EI} = \dfrac{20\text{x}10^9}{(2.0\text{x}10^5)(1.067\text{x}10^9)}$ radians

Slope $\theta_B = 9.37 \times 10^{-5}$ radians

(iii) Deflection (δ_B)

By Mohr's Theorem – II

Deflection $\delta_B = \dfrac{A\bar{x}}{EI}$

Deflection $\delta_{max} = \dfrac{1}{EI}(A\,\bar{x}) = \dfrac{1}{(2.0\text{x}10^5)(1.067\text{x}10^9)}$ (20 x 1.33) x10^{12}

= 1.24×10^{-4} m

δ_{max} =0.125mm

Alternate method

(i)Slope θ = $\frac{W\ell^2}{2EI}$ = $\frac{10 \text{x} 10^3 \text{x} (2000)^2}{2 \text{x} \left[(2\text{x}10^5)(1.067\text{x}10^9) \right]}$ = 9.37 x 10^{-5} radian

(ii) Deflection (δ_B)

δ_B = $\frac{W\ell^3}{3EI}$ = $\frac{10 \text{x} 10^3 \text{x} (2000)^3}{3\left[(2\text{x}10^5)(1.067\text{x}10^9) \right]}$ = 0.124mm

= 0.125mm

Result

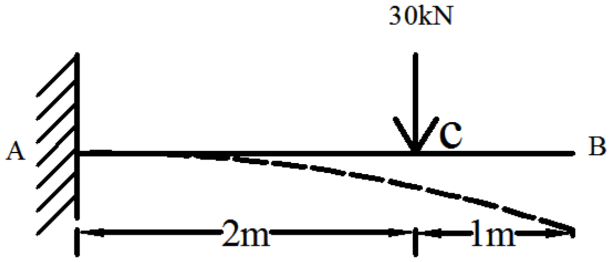
θ_B = 9.37 x 10^{-5} radians

δ_B = 0.125mm

Problem 2

A cantilever beam 3m span carries a point load of 30 kN at 1m from free end. Find max. Slope and deflection. Take EI = 10 x 10^4 kN.m^2

Given data:



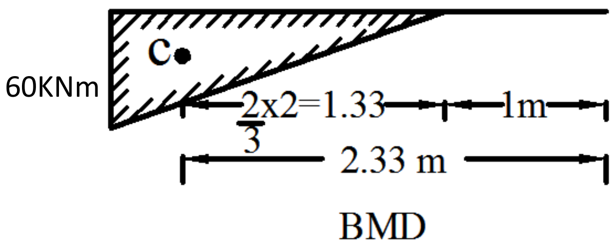
Span ℓ = 3m

W = 30 KN

l_1 = 2 m

EI = 10 x 10^4 kN.m^2

Draw BMD as shown in fig.1.10



Solution

Draw BMD

Let A = area of BMD

A = $\frac{1}{2}$xbxh

Bending Moment @ A= 30x 2 = 60 kNm

A = $\frac{1}{2}$ x 2 x 60 = 60kNm^2

$\bar{x}_1$ = (1 + $\frac{2}{3}$ x 2) = 2.33 m

(i) Slope

By Mohr's Theorem - I

$\theta_B = \dfrac{A}{EI} = \dfrac{60}{10 \times 10^4}$ radians

$\theta_B = 6 \times 10^{-4}$ radians

(ii) Deflection

By Mohr's Theorem – II

$\delta_{max} = \dfrac{A\overline{x_1}}{EI}$

$\delta_B = \dfrac{A\overline{x_1}}{EI} = \dfrac{60 \times 2.33}{10 \times 10^4}$ $=1.398 \times 10^{-3}$m$=1.398$ mm = Deflection at B

Problem 3

A cantilever beam of span 4m is subjected to a udl of 20 kN/m over a entire length. Find the maximum slope and deflection.Take E = 2.1 x 10^5 N/mm^2, I = 15 x 10^8mm^4

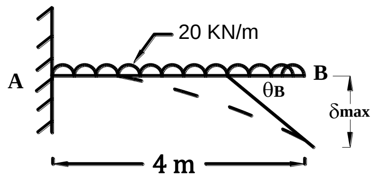
Given data:

Span ℓ = 4m

Udl w = 20 kN/m

E = 2.1 x 10^5 N/mm^2

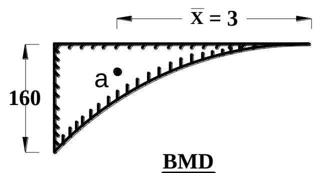
I = 15 x 10^8mm^4



Required

Slope at θ_B = ?

Deflection at B δ_B = ?



BMD

(i) Bending Moment

B.M @ A due to udl load $= \dfrac{w\ell^2}{2} = \dfrac{20 \times 4^2}{2}$ = 160 KN.m

Draw BMD

A = area BMD

A $= \dfrac{1}{3}$xbxh

A $= \dfrac{1}{3}$ x 4 x 160 = 213.33 kN.m^2= 213.33x10^9 N.mm^2

$\bar{x}$ = Centroid from free end

$\bar{x} = \dfrac{3}{4} \times l = (\dfrac{3}{4} \times 4) = 3$ m $=3 \times 10^3$ mm

(ii) Slope

By Mohr's Theorem - I

$$\theta_B = \frac{A}{EI} = \frac{213.34x10^9}{(2.1x10^5)(15x10^8)} \text{ radians}$$

θ_B = 6.772 x 10^{-4} radians

(iii) Deflection

By Mohr's Theorem – II

$$\delta_{max} = \frac{A\bar{x}}{EI} = \frac{(213.34x3)x10^{12}}{(2.1x10^5)(15x10^8)} = 2.03 \text{ mm}$$

Result

θ_B = 6.772 x 10^{-4} radians

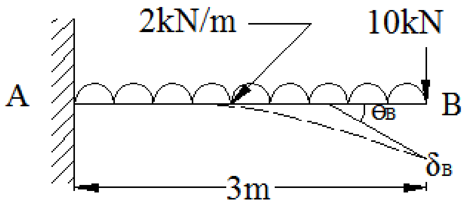
δ_{max} = 2.03 mm

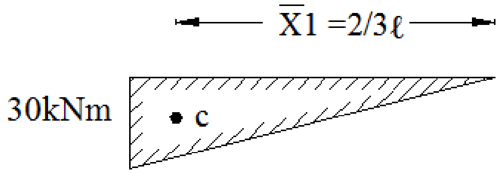
Problem 4

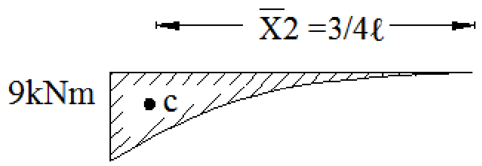
A cantilever beam 3m span carries a point loads of 10 kN at free end and an udl of 2 kN/m over its entire span. Find the max. slope and deflection. Take EI = 2 x 10^4 kN.m^2

Given data:

Span	ℓ	= 3m
Point load	W	= 10 kN
udl	w	= 2 kN/m
	EI	= 2 x 10^4 kN.m^2







Solution:

(i) Bending Moment

Bending Moment @ A due to point load

$= Wl = 10 \times 3 = 30 \text{kN.m}$

Bending Moment @ A due to udl

$$= \frac{wl^2}{2} = \frac{2 \times 3^2}{2} = 9 \text{kN.m}$$

Draw BMD

A_1 = area of BMD for point load

A_2 = area of BMD for udl

Due to Point load

$$A_1 = \frac{1}{2} \text{xbxh}$$

$$A_1 = \frac{1}{2} \text{ x 3 x 30} = 45 \text{ kNm}^2$$

$$\bar{x}_1 = \frac{2}{3} \text{ x 3 = 2m}$$

Due to Udl

$A_2 = \frac{1}{3} \times b \times h$

$A_2 = \frac{1}{3} \times 3 \times 9 = 9\ kNm^2$

$\overline{x_2} = \frac{3}{4} \times 3 = 2.25m$

(ii) Slope

By Mohr's Theorem - I

$\theta_B = \frac{A}{EI} = \left(\frac{A_1 + A_2}{EI}\right)$ radian

$\theta_B = \frac{1}{2 \times 10^4}[45+9] = 2.7 \times 10^{-3}$ radians

(iii) Deflection

By Mohr's Theorem – II

$\delta_{max} = \delta_B = \frac{A\bar{x}}{EI} = \frac{1}{EI}(A_1 \bar{x}_1 + A_2 \bar{x}_2)$

$= \frac{1}{2 \times 10^4}((45 \times 2) + (9 \times 2.25))$

$\delta_{max} = 5.513 \times 10^{-3} m = 5.513mm$

Result

$\theta_B = 2.7 \times 10^{-3}$ radians

$\delta_B = 5.513mm$

Problem 5

A Cantilever beam of span 5m is carrying a point load of 16KN at 4m from fixed end. Calculate the slope and deflection at load point and at the free end by area moment theorem. Take $E = 1.5 \times 10^5\ N/mm^2$ and $I = 4 \times 10^8\ mm^4$.

Given data:

Cantilever beam with point load

Point load $W = 16KN$

Span $\ell = 5m$

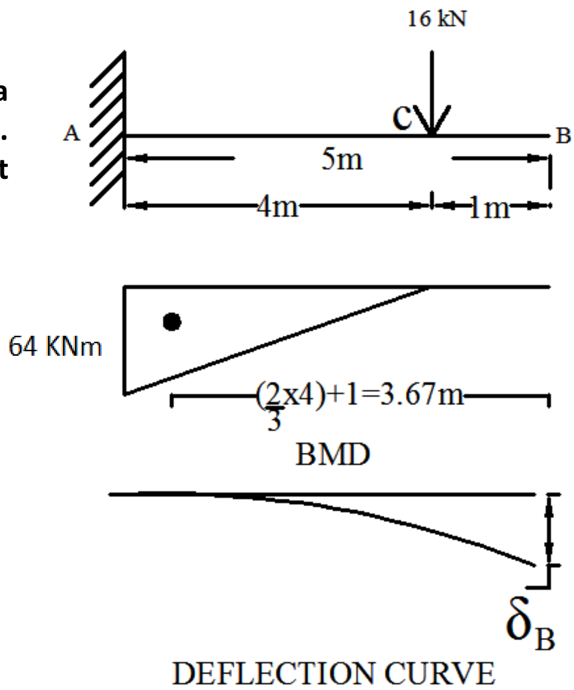
$E = 1.5 \times 10^5\ N/mm^2$

$I = 4 \times 10^8\ mm^4$

To Find:

Max slope = θ = ?

Max deflection = δ = ?



Solution:

Area of BMD $A = \frac{1}{2} \times bh$

Bending moment $= WL$

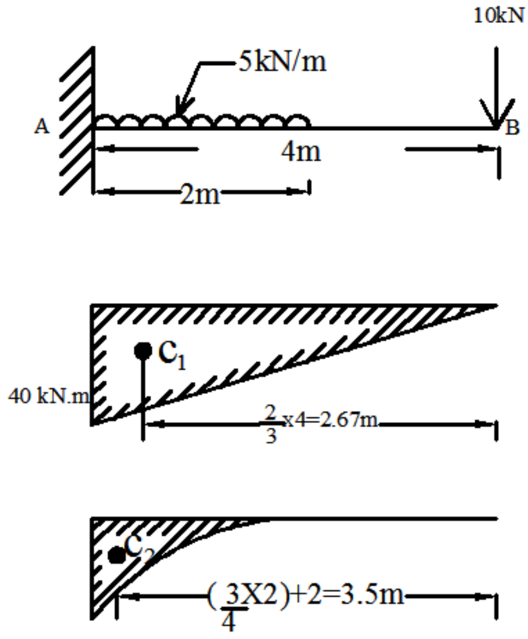
$= 16 \times 4 = 64$ kNm

$A = \frac{1}{2} \times b \times h$

$A = \frac{1}{2} \times 4 \times 64$

$A = 128 \text{ kN.m}^2$

$A = 128 \times 10^9 \text{ N.mm}^2$



BMD

Centroid of BMD

$\bar{x} = (\frac{2}{3} \times 4) + 1$

$= 3.67\text{m} = 3.67 \times 10^3$ mm

Applying Mohr's theorem - I

(i) Slope $= \theta_B = \theta_{max} = \dfrac{A}{EI}$

$$= \frac{128 \times 10^9}{(1.5 \times 10^5)(4 \times 10^8)}$$

$\theta_B = \theta_{max} = 2.13 \times 10^{-3}$ Radians

Applying Mohr's theorem - II

(ii) Deflection $= \delta_B = \delta_{max} = \dfrac{A\bar{x}}{EI}$

$$= \frac{(128 \times 3.67) \times 10^{12}}{(1.5 \times 10^5)(4 \times 10^8)}$$

Deflection $= \delta_B = \delta_{max} = 7.83$mm

Result:

Slope $= \theta_B = \theta_{max} = 2.13 \times 10^{-3}$ Radians

Deflection $= \delta_B = \delta_{max} = 7.83$mm

Problem 6

A cantilever beam 4m span carries an udl of 5 kN/m over 2m from fixed end and a point load of 10 kN at free end. Find the max. slope and deflection. Take EI = 10 x 10^4 kN.m^2.

Given Data:

Span	ℓ	= 4m
udl	w	= 5 kN/m
Point load	W	= 10 kN

EI $= 10 \times 10^4$ kN.m^2

x = 2 m

Solution

(i) Bending Moment

BM due to point load w. ℓ

$= 10 \times 4 = 40$ kN.m

BM due to udl at A= $\frac{wx^2}{2}$

$= 5 \times 2 \times 2/2 = 10$ kN.m

Draw BMD by parts as shown in fig.1.13

Due to point load,

$A_1 = \frac{1}{2} \times b \times h$

$A_1 = \frac{1}{2} \times 4 \times 40 = 80\text{kNm}^2$

$\overline{x_1} = \frac{2}{3} \times 4 = 2.67$ m

Due to UDL,

$A_2 = \frac{1}{3} \times b \times h$

$A_2 = \frac{1}{3} \times 2 \times 10 = 6.67\ \text{kNm}^2$

$\overline{x_2} = \left(2 + \frac{3}{4} x 2\right) = 3.5\text{m}$

(ii) Slope

By Mohr's Theorem - I

$\theta_B = \frac{A}{EI} = \frac{(A_1 + A_2)}{EI} = \frac{80 + 6.67}{10\,x10^4} = 8.67\text{x}\ 10^{-4}$ radians

(iii) Deflection

By Mohr's Theorem – II

$\delta_{max} = \delta_B = \frac{A\bar{x}}{EI} = \frac{1}{EI}\ (A_1\ \bar{x}_1 + A_2\ \bar{x}_2) = \frac{1}{10\,x10^4}\ (80\text{x}2.67 + 6.67\text{x}3.5)$

$\delta_{max} = \delta_B = 2.37 \times 10^{-3}$ m = 2.37mm

Result

$\theta_B = 8.66 \times 10^{-4}$ radians

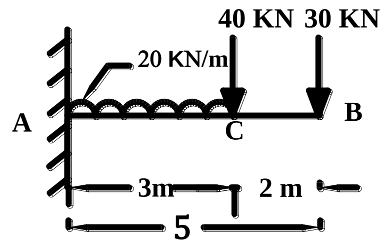
$\delta_{max} = 2.37$mm

Problem 7

A cantilever beam 5m span carries an udl of 20 kN/m over a length of 3 m from fixed end and two point loads of 40 kN and 30 kN at 3m and 5m from the fixed end respectively. Determine the maximum slope and deflection at the free end using Mohr's Theorem. Take EI = 47.05 x 10^3 kN.m^2.

Given data:

`
Span ℓ = 5m
udl w = 20 kN/m
x = 3 m
W = 30 kN at free end
W_1 = 40 kN at ℓ_1= 3 m
EI = 47.05 x 10^3 kN.m^2



Required

θ_B = ? δ_B =?

Solution:

(i) Bending Moment

Bending Moment at A due to udl

$$= \frac{wx^2}{2} = \frac{20 \times 3^2}{2} = 90\,\text{kN.m}$$

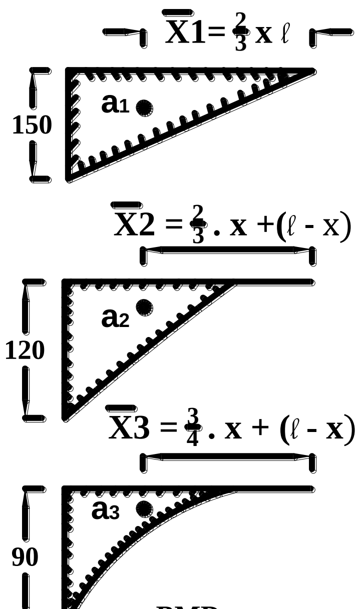
Bending Moment at A due to load W

W x ℓ = 30 x 5 = 150 kN.m

Bending Moment at A due to load W_1

W . X = 40 x 3 = 120 kN.m

Draw BMD



BMD

$A_1 = \frac{1}{2}$ xbxh

$A_1 = \frac{1}{2}$ x 5 x 150 = 375 kN.m^2

$\overline{x_1} = \frac{2}{3} x\, l = \frac{2}{3}$ x 5 = 3.33m

$A_2 = \frac{1}{2}$ x bxh

$A_2 = \frac{1}{2}$ x 3 x 120 = 180 kN.m^2

$\overline{x_2} = \frac{2}{3} x + (\ell - x) = (\frac{2}{3}$ x 3) + 2 = 4m

$A_3 = \frac{1}{3}$ x b x h

$A_3 = \frac{1}{3}$ x 3 x 90 = 90 kN.m^2

$\overline{x_3} = (\frac{3}{4} x) + (\ell - x) = (\frac{3}{4}$ x 3) + 2 = 4.25m

(ii)Slope

By Mohr's Theorem - I

$\theta_B = \frac{A}{EI} = \left(\frac{A_1 + A_2 + A_3}{EI}\right)$ radian

$\theta_B = \frac{1}{47.05 x 10^3}(375+180+90) = 13.7088 x 10^{-3}$ radians

= 0.0137 radians

(iii)Deflection

By Mohr's Theorem – II

$\delta_{max} = \delta_B = \frac{A\bar{x}}{EI} = \frac{1}{EI}$ $(A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3)$

$= \frac{1}{47.05 x 10^3}$ [(375x3.33) + (180x4)+(90x4.25)]

$\delta_{max} = 49.97 \text{ x } 10^{-3}$m = 49.97mm

Result

$\theta_B = 13.7088 x 10^{-3}$ radians

δ_B = 49.97 mm

Problem 8

A cantilever 2m long carries a point load of 9 kN at free end and a udl of 8 kN/m. Over a length of 1m from the fixed end. Determine the deflection at the free end if E = 2 x 10^5 N/mm^2. Take I = 2250 x 10^4 mm^4.

Given data:

E = 2 x 10^5 N/mm^2

I = 2250 x 10^4mm^4

Span ℓ = 2m

Point load W = 9 kN;

Udl w = 8 kN/m

Solution

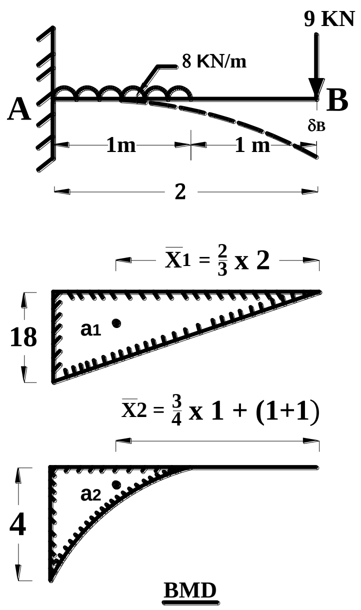
(i) Bending Moment

B.M. due to point load = Wx l

= 9 x 2 = 18 KN.m

B.M. due to udl load = $\frac{wl^2}{2}$

= 8x1x $\frac{1}{2}$ = 4 KN.m



Draw BMD

$A_1 = \frac{1}{2}$ x b x h

$A_1 = \frac{1}{2} \times 2 \times 18 = 18 \text{ kNm}^2$

$= 18 \times 10^9 \text{ Nmm}^2$

$A_2 = \frac{1}{3} \times b \times h$

$A_2 = \frac{1}{3} \times 1 \times 4 = 1.33 \text{ kNm}^2$

$= 1.33 \times 10^9 \text{ Nmm}^2$

$\overline{x_1} = \frac{2}{3} \times 2 = 1.33 \text{ m} = 1.33 \times 10^3 \text{ mm}$

$\overline{x_2} = 1 + (\frac{3}{4} \times 1) = 1.75 \text{ m} = 1.75 \times 10^3 \text{ mm}$

By Mohr's Theorem - I

(ii) Slope

$$\theta_B = \frac{A}{EI} = \frac{1}{EI}(A_1 + A_2) = \frac{1}{(2\text{x}10^5)(2250\text{x}10^4)}(18 + 1.33)\text{x}10^9$$

$\theta_B = 4.30 \times 10^{-3}$ radians

By Mohr's Theorem - II

(iv) Deflection

$$\delta_{max} = \frac{A\bar{x}}{EI}$$

$$= \frac{1}{EI}(A_1 \bar{x}_1 + A_2 \bar{x}_2)$$

$$= \frac{1}{(2\text{x}10^5)(2250\text{x}10^4)}[(18 \times 1.33) + (1.33 \times 1.75)] \times 10^{12}$$

$\delta_{max} = 5.84$ mm

Result

$\theta_B = 4.30 \times 10^{-3}$ radians

$\delta_{max} = 5.84$ mm

Problem 9

A cantilever beam of span 4m is subjected to a udl of 20 kN/m over a entire length and a point load of 15 kN acting at the centre. Find the maximum deflection for the beam. Given E = 2.1 x 10^5 N/mm^2, I = 15 x 10^8mm^4

Given data:

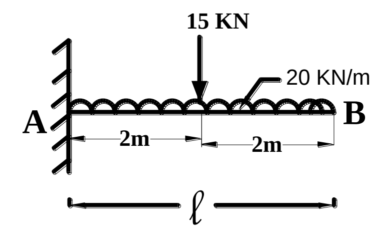
Span	ℓ	= 4m
Udl	w	= 20 kN/m
Point load	W	= 15 kN
	E	= 2.1 x 10^5 N/mm^2
	I	= 15 x 10^8mm^4

Solution

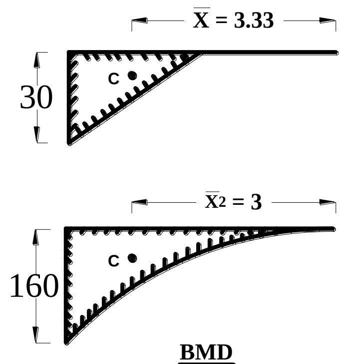
(i) Bending Moment

B.M. due to point load Wx l= 15 x 2 = 30 KN.m

B.M. due to udl load $\frac{wl^2}{2} = \frac{20x4^2}{2}$ = 160 KN.m



Draw BMD



BMD

$A_1 \quad = \frac{1}{2}$ x b x h

$A_1 \quad = \frac{1}{2}$ x 2 x 30 = 30 kNm^2

=30x10^9 Nmm^2

$\overline{x_1} \quad = (2 + \frac{2}{3} \text{ x } 2)$ = 3.33 m= 3.3 x 10^3 mm

$A_2 \quad = \frac{1}{3}$ x bx h

$A_2 \quad = \frac{1}{3}$ x 4 x 160 = 213.33 kNm^2

=213.33x10^9 Nmm^2

$\overline{x_2} \quad = \frac{3}{4}$ x 4 = 3m=3 x 10^3 mm

By Mohr's Theorem – II

$$\delta_{max} = \frac{A\overline{x}}{EI} = \frac{1}{EI}(A_1\,\overline{x}_1 + A_2\,\overline{x}_2)$$

$$= \frac{1}{(2.1\text{x}10^5)(15\text{x}10^8)} [(30 \text{ x}3.3) + (213.34\text{x}30)]\text{x}10^{12}$$

δ_{max} = 2.35mm

Result

δ_{max} = 2.35mm

Problem 10

A cantilever beam 3m long carries an udl of w/m over its entire span. The size of beam is 75mm x 150mm. If the max. deflection is 2.5mm, determine the load w/m. Take E = 2 x 10^5 N/mm^2. Also find the max slope.

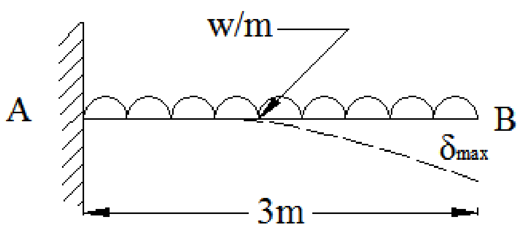
Given data:

b = 75mm

d = 150mm

δ_B = 2.5mm = 2.5 x 10^{-3} m

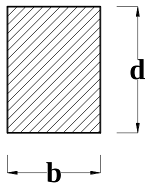
E = 2 x 10^5 N/mm^2



Solution:

(i). Moment of inertia (I)

$$I = \frac{bd^3}{12} = \frac{75 x 150^3}{12} = 21.09 \times 10^6 mm^4$$



(ii). Total udl/m (w)

We know, $\delta_B = \frac{w\ell^4}{8EI} = 2.5$ mm

$$\frac{w\ell^4}{8EI} = \delta_B$$

$$W = \frac{\delta_B \text{ x } 8\,\text{EI}}{l^4} = \frac{2.5 \text{ x } 8\,((2\text{x}10^5)(21.09\text{x}10^6))}{3000^4}$$

w = 1.04 N/mm or kN/m

(iii) Slope (θ)

$$\text{Slope} = \frac{w\ell^3}{6EI} = \frac{1.04 \text{ x } 3000^3}{6 \text{ x } (2\text{x}10^5)(21.09\text{x}10^6)} \text{ radian} = 1.109 \times 10^{-3} \text{ radians}$$

Result

W = 1.04 kN/m

θ_B= 1.109 x 10^{-3} radians

Problem 11

A cantilever beam of 1m long of rectangular section of width 40mm and depth 60mm. Calculate the maximum udl that can be allowed over the entire length of the beam without exceeding a deflection of 3.5 mm at the free end. Also calculate the maximum slope at the free end. Take E = 7 x 10^4 N/mm^2.

Given data:

Span	ℓ	= 1m=1000mm
Width	b	= 40 mm
depth	d	= 60 mm
δ_{max}		=3.5 mm
E		= 7 x 10^4 N/mm^2

Solution

(i) Moment of inertia (I)

$$\text{M.I. (I)} = \frac{bd^3}{12} = \frac{40 x 60^3}{12} = 720 \times 10^3 mm^4$$

(ii) Safe udl (w/m)

$\delta_{max} = \delta_B = 3.5$ mm

Cantilever beam with udl = $\delta_{max} = \frac{w\ell^4}{8EI}$

$$\frac{w\ell^4}{8EI} = \delta_{max}$$

$$\frac{\text{wx}1000^4}{8\text{x}(7\text{x}10^4)(720x10^3)} = 3.5 \text{ mm}$$

w = 1.4112 N/mm or KN/m

Safe udl (w) = 1.4112 KN/m.

iii) Maximum Slope

θ_B = Maximum slope at B

$$\theta_B = \frac{w\ell^3}{6EI} = \frac{1.4112 \text{ x } 1000^3}{6 \text{ x } (7\text{x}10^4)(720\text{x}10^3)} \text{ radians}$$

$\theta_B = 4.67 \text{ x } 10^{-3}$ radian

Result

Safe udl (w) = 1.4112 KN/m.

$\theta_B = 4.67 \text{ x } 10^{-3}$ radians

1.1.7 SIMPLY SUPPORTED BEAM WITH CENTRAL POINT LOAD(W)

BOOK WORK – 6

SLOPE and DEFLECTION FOR SIMPLY SUPPORTED BEAM WITH CENTRAL POINT LOAD

SOLUTION

Consider simply supported beam loaded as shown in fig.

$$A = \frac{1}{2} \text{ x b x h}$$

$$A = \frac{1}{2} \text{ x } \frac{l}{2} \text{ x } \frac{W\ell}{4} = \frac{W\ell^2}{16}$$

$$\bar{x} = \frac{2}{3} \text{ x } \frac{l}{2} = \frac{l}{3} = \text{Centroid from A}$$

i) Slope

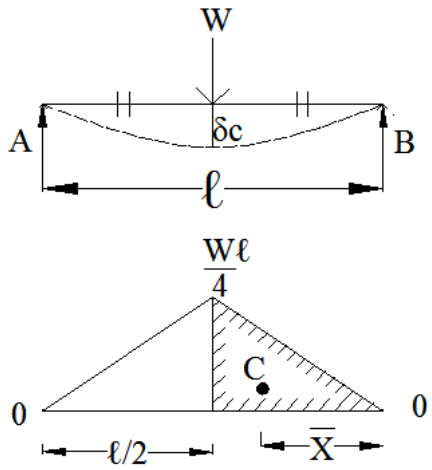
By Mohr's Theorem - I

$$\theta_{max} = \frac{A}{EI} = \frac{1}{\text{EI}} \text{ x A}$$

$$\theta_{max} = \frac{1}{EI}\left(\frac{W\ell^2}{16}\right) = \frac{W\ell^2}{16\,EI}$$

$$\boxed{\theta_{max} = \frac{W\ell^2}{16\,EI}}$$

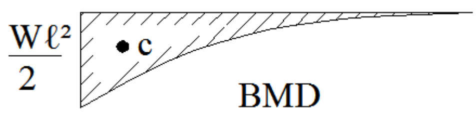
It is symmetrically loaded $\theta_A = \theta_B = \theta_{max} = \frac{W\ell^2}{16\,EI}$ `



By Mohr's Theorem – II

$$\delta_c = \frac{A\bar{x}}{EI} = \frac{1}{EI}\left(\frac{W\ell^2}{16} \text{x} \frac{l}{3}\right) = \frac{W\ell^3}{48\,EI}$$

$$\boxed{\delta_c = \delta_{max} = \frac{W\ell^3}{48\,EI}}$$



1.1.8 Book Work – 7

Slope and Deflection for simply supported beam with udl over the entire span

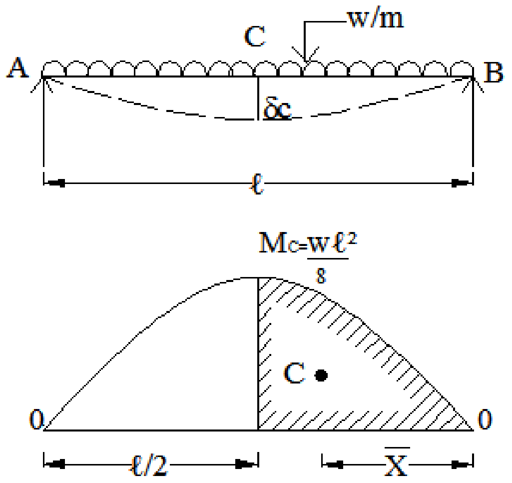
Solution

Consider simply supported beam loaded as shown in fig.

Draw BMD

Consider two points B & C

Draw tangents at B & C



Let A $= \frac{2}{3} \text{ x bx h}$

A $= \frac{2}{3} \times \frac{l}{2} \times \frac{w\ell^2}{8} = \frac{w\ell^3}{24}$

$\bar{x} = \frac{5}{8} \times \frac{l}{2} = \frac{5\ell}{16}$

(i) Slope

By Mohr's Theorem - I

$\theta_{A/C} = \frac{A}{EI} = \frac{1}{EI}\left(\frac{w\ell^3}{24}\right) = \frac{w\ell^3}{24EI}$

Since it is symmetrically loaded

$\theta_A = \theta_B = \frac{w\ell^3}{24EI}$

(ii) Deflection

By Mohr's Theorem – II

$\delta_c = \frac{A\bar{x}}{EI} = \frac{1}{EI}\left(\frac{w\ell^3}{24} \times \frac{5\ell}{16}\right)$

$\delta_c = \delta_{max} = \frac{5w\ell^4}{384EI}$

Problem 12

A Simply supported beam 5m span is 200mm x 300mm of size . It carries an UDL of 5KN/ m over the entire span. Calculate the Max slope & deflection by area moment method. Take E = 1.2 x 10^5 N / mm^2.

Given data:

Udl w = 5KN/m

Span ℓ = 5m

Breadth b = 200mm

Depth d = 300mm

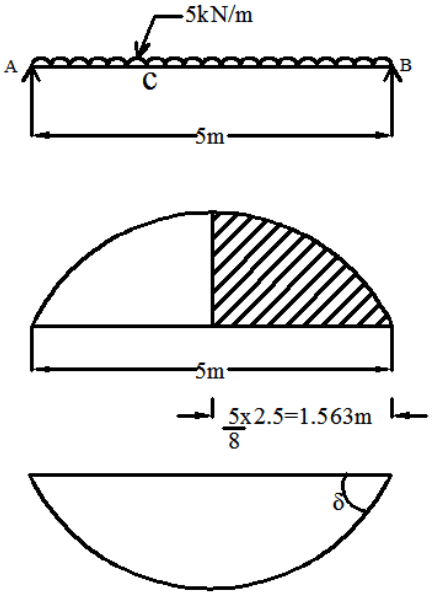
E = 1.2 x10^5 N/mm^2

To find

Max Slope = θ =?

Max deflection = δ =?

Solution:



DEFLECTION

Area of BMD

$$A = \frac{2}{3} \text{ x bh}$$

$$\text{Bending moment} = \frac{wl^2}{8}$$

$$= \frac{5x5^2}{8}$$

$$= 15.625 \text{ kN . m}$$

$$A = \frac{2}{3}\text{x2.5x15.625}$$

$$= 26.042 \text{ KN. m}^2$$

$$A = 26.042\text{x}10^9 \text{ N.mm}^2$$

Centriod of BMD

$$\bar{x} = \left(\frac{5}{8}\text{x b}\right)$$

$$\bar{x} = \left(\frac{5}{8}\text{x2.5}\right)$$

$$\bar{x} = 1.563\text{m} = 1563 \text{ mm}$$

$$\text{Moment of inertia } I = \frac{bd^3}{12}$$

$$= \frac{200\text{x}300^3}{12}$$

$$I = 450 \text{ x } 10^6 \text{ mm}^4.$$

Applying Mohr 's theorem – I

(i) Slope = $\theta_B = \theta_{max} = \frac{A}{EI}$

$$= \frac{(26.042\text{x}10^9)}{(1.2\text{x}10^5)(450\text{x}10^6)}$$

Slope = $\theta_B = \theta_{max} = 4.82 \text{ x } 10^{-4}$ Radians.

Applying Mohr 's theorem – II

(ii) Deflection = $\delta_B = \delta_{max} = \frac{A\bar{x}}{EI}$

$$= \frac{(26.042\text{x}1.563)\text{x}10^{12}}{(1.2\text{x}10^5)(450\text{x}10^6)}$$

Deflection = $\delta_B = \delta_{max} = 0.75$mm.

Result:

Slope = $\theta_B = \theta_{max}$ = 4.82 x 10^{-4} Radians.

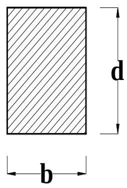
Deflection = $\delta_B = \delta_{max}$ = 0.75mm

Problem 13

A simply supported beam 6m long, 150mm x 300mm size carries a central point load of 40 kN. Determine the maximum slope and deflection. Take E = 1.5 x 10^5 N/mm^2.

Given data:

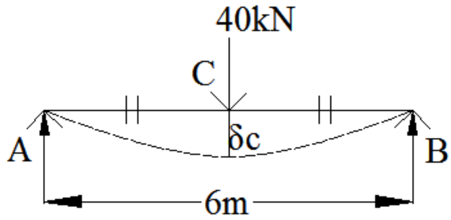
Span	ℓ	= 6m
Load	W	= 40 kN
Wide	b	= 150mm
Depth	d	= 300mm
E		= 1.5 x 10^5 N/mm^2



Solution:

(i) Moment of inertia (I)

$$I = \frac{bd^3}{12} = \frac{150*300^3}{12} = 337.5 \times 10^6 \text{ mm}^4$$



Bending Moment:

$$\text{Max. B.M.} = \frac{W\ell}{4} = \frac{40*6}{4} = 60 \text{ kN.m}$$

Area of BMD

A = area of BMD between B & C

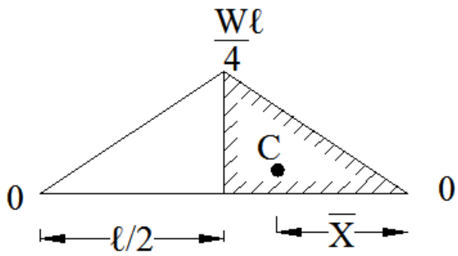
$\bar{x}$ = Centroid from B

$$A = \frac{1}{2} \times b \times h$$

$$A = \frac{1}{2} \times 3 \times 60 = 90 \text{ kNm}^2$$

$$= 90\times10^9 \text{ Nmm}^2$$

$$\bar{x} = \frac{2}{3} \times \frac{6}{2} = 2m = 2\times10^3 \text{ mm}$$



BMD

(ii) Slope (θ)

By Mohr's theorem – I

$$\theta_A = \theta_B = \frac{A}{EI} = \frac{90\times10^9}{(1.5\times10^5)(337.5\times10^6)} \text{ radians}$$

$\theta_A = \theta_B$ = 1.78 x 10^{-3} radians

(iii) Deflection (δ_{max})

By Mohr's theorem – II

$$\delta_C = \frac{A\bar{x}}{EI} = \frac{1}{EI}\left(A\bar{x}\right)$$

$$\delta_c = \frac{1}{(1.5x10^5)(337.5x10^6)}(90x\ 2)x10^{12} = 3.56mm$$

Result

θ_{max} = 1.78 x 10^{-3} radians

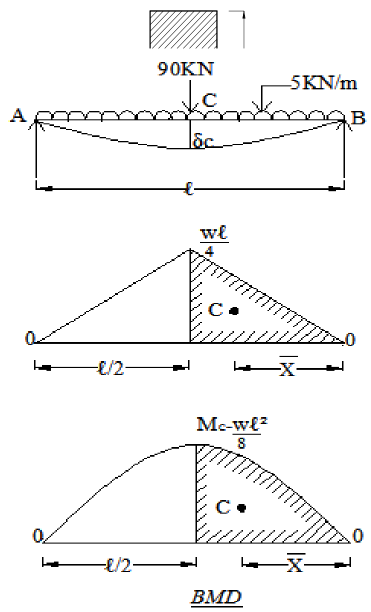
δ_c = 3.56mm

Problem 14

A simply supported beam 8m long carries a point load of 90 kN at centre and udl of 5 kN/m over at entire span. The size of beam is 200mm x 400mm E = 1.5 x 10^4 N/mm². Determine the maximum slope and deflection.

Given data:

Point load W = 90 kN
Udl w = 5 kN/m
Wide b = 200mm
Depth d = 400mm
E = 1.5 x 10^4 N/mm²



Required:

Slope & Deflection

Solution:

i) Moment of inertia(I)

$$M.I. = I = \frac{bd^3}{12} = \frac{200\,x\,(400)^3}{12} = 1.067 \text{ x } 10^9 \text{ mm}^4$$

By Area Moment Method

ii) Free BMD

$$\text{BM due to Point load} = \frac{W\ell}{4} = \frac{90\,x\,8}{4} = 180 \text{ kN.m}$$

$$\text{BM due to udl} = \frac{w\ell^2}{8} = \frac{5\,x\,8^2}{8} = 40 \text{ kN.m}$$

iii) Area of BMD

$$\text{Area of triangle } A_1 = \frac{1}{2} \text{ x b x h}$$

$$A_1 = \frac{1}{2}\text{x4x180} = 360 \text{ kN.m}^2$$

$$= 360x10^9 \text{ Nmm}^2$$

$$\text{Centroid } \overline{x_1} = \frac{2}{3} \text{ x 4} = 2.67 \text{ m} = 2.67x10^3 \text{ mm from B}$$

$$\text{Area of Parabola } A_2 = \frac{2}{3} \text{ x b x h}$$

$$A_2 = \frac{2}{3}\text{x4x40} = 106.67 \text{ kN.m}^2$$

$$= 106.67x10^9 \text{ Nmm}^2$$

Centroid $\overline{x_2}$ = $\frac{5}{8}$ x 4 = 2.5 m= 2.5x10^3 mm

(iv) Slope (θ)

By Mohr's theorem – I

$\theta_A = \theta_B = \frac{A}{EI} = \frac{1}{EI}[A_1 + A_2]$

$= \frac{1}{(1.5x10^4)(1.067x10^9)}[360+106.67]x10^9$ radians

θ_A = 29.16 x 10^{-3} radians = 0.0291 radians

(v) Deflection (δ_{max})

By Mohr's theorem – II

$\delta_{max} = \frac{A\overline{x}}{EI} = \frac{1}{EI}\left(A_1\overline{x_1} + A_2\overline{x_2}\right)$

$\delta_{max} = \frac{1}{(1.5x10^4)(1.067x10^9)}\left[(360\ x2.67)+(106.67\ x2.5)\right]x10^{12}$

δ_{max}= 76.7 mm

Result

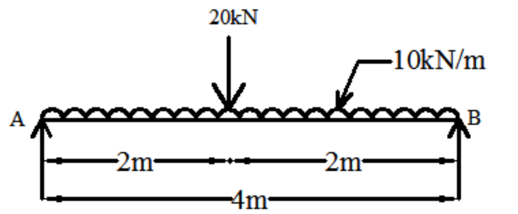
$\theta_A = \theta_B$ = 0.0291 radians

$\delta_c = \delta_{max}$ = 0.0767m

Problem 15

A Simply supported beam of span 4 m carries an UDL of 10KN/m over the full length and a central point load of 20KN. determine the maximum slope and maximum deflection by area moment method.

Take E = 2x 10^5 N /mm^2 and I = 8x10^7 mm^4.



Given data:

S.S.B with UDL & P.L

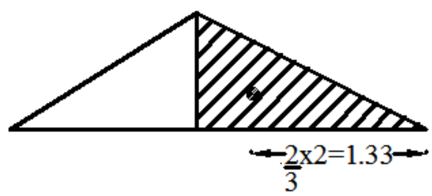
UDL (w) = 10 KN/ m

P.L (W) = 20 KN

Span ℓ = 4m

E = 2x10^5 N/mm^2

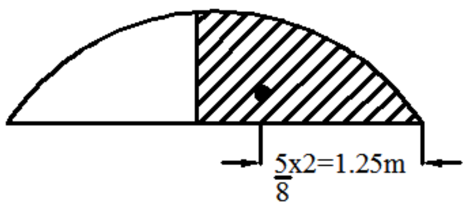
I = 8x 10^7 mm^4



To Find:

Slope = θ = ?

Deflection = δ =?



Solution:

(i) Area of BMD (Point load)

Area of BMD $A_1 = \frac{1}{2}$ x bh

Bending moment = $\frac{Wl}{4}$

$= \frac{20x4}{4}$

= 20KN.m

$A_1 = \frac{1}{2}$ x 2 x20

= 20 KN .m^2

A_1 = 20 x 10^9 N.mm^2

(ii) Area of BMD (UDL)

Area of BMD **$A_2 = \frac{2}{3}$ x bh**

Bending moment $= \frac{Wl^2}{8}$

$= \frac{10x4^2}{8}$

= 20 kN.m

$A_2 = \frac{2}{3}$ x2 x 20

= 26.67 KN.m^2

A_2 = 26.67 x 10^9 N .mm^2

Centriod of BMD (P.L)

$\bar{x}_1 = \frac{2}{3} x\ell$

$= \frac{2}{3}$ X 2

= 1.33m

$\bar{x}_1$ =1.33x10^3 mm.

Centriod of BMD (UDL)

$\bar{x}_2 = \frac{5}{8} x\ell$

$= \frac{5}{8}$ x 2 = 1.25m

$\bar{x}_2$ = 1.25x10^3 mm.

Applying Mohr's theorem - I

(iii) Slope = θ_B =θ_{max} = $\frac{(A_1 + A_2)}{EI}$

$= \frac{(20+26.67)x10^9}{(2x10^5)(8x10^7)}$

Slope = θ_B =θ_{max} = 2.92X10^{-3} Radians.

Applying Mohr's theorem - II

(iv) Deflection = $\delta_B = \delta_{max}$ = $\dfrac{(A_1\overline{x_1} + A_2\overline{x_2})}{EI}$

$= \dfrac{[(20x1.33)+(26.67x1.25)]x10^{12}}{(2x10^5)(8x10^7)}$

Deflection = $\delta_B = \delta_{max}$ = 3.75 mm.

Result:

Slope = $\theta_B = \theta_{max}$ = 2.92X10^{-3} Radians.

Deflection = $\delta_B = \delta_{max}$ = 3.75 mm.

Problem 16

A Simply Supported beam 6m span carries two point loads of 15 kN each at one third span. Determine the maximum slope and deflection.
Take EI = 2 x 10^4 kN.m^2.

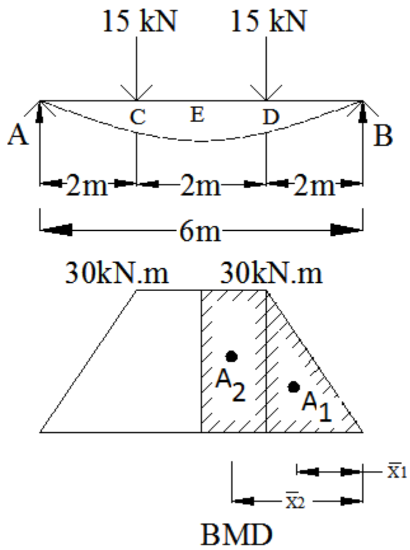
Given data:

W = 15 KN

ℓ = 6m

A = $\dfrac{l}{3}$ = $\dfrac{6}{3}$ = 2m

EI = 2 x 10^4 KN.m^2



Solution

(i) Bending moment

Since the load is symmetrical

$R_A = R_B = \dfrac{\text{Total load}}{2} = \dfrac{15+15}{2}$ = 15 KN

$M_A = M_B = 0$

$M_C = M_D$ = 15 x 2 = 30 KN

(ii) Area of BMD

Area of triangle A1 = $\dfrac{1}{2}$ x b x h

A1 = $\dfrac{1}{2}$ x2x30 = 30 kN.m^2

Centroid $\overline{x_1}$ = $\dfrac{2}{3}$ x 2 = 1.33 m from B

Area of rectangle A_2 = b x h

A_2 = 1 x 30= 30 kN.m^2

Centroid $\overline{x_2}$ = 2+$\dfrac{1}{2}$ = 2.5 m

(iii) Slope (θ)

By Mohr's theorem – I

$\theta_A = \theta_B = \frac{A}{EI} = \frac{1}{EI}[A_1+A_2]$

$= \frac{1}{2*10^4}[30+30]$ radian

$\boldsymbol{\theta_B = 3 \times 10^{-3}}$ **radian**

(iv) Deflection (δ_{max})

By Mohr's theorem – II

$\delta_{max} = \frac{A\bar{x}}{EI} = \frac{1}{EI}\left(A_1\bar{x}_1 + A_2\bar{x}_2\right)$

$\delta_{max} = \frac{1}{2x10^4}$ [(30 x 1.33) + (30 x 2.5)]

$= 5.745 \times 10^{-3}$ m

$\boldsymbol{\delta_{max}}$**= 5.745 mm**

Result:

Slope $\boldsymbol{\theta_B = 3 \times 10^{-3}}$ **radian**

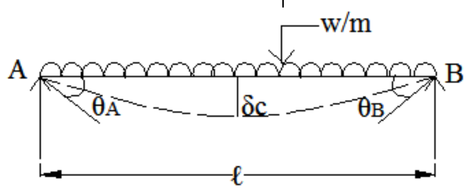
Deflection $\boldsymbol{\delta_{max}}$**= 5.745 mm**

Problem 17

A girder of uniform section and constant depth 1800mm is freely supported over a span of 20 meter. Calculate the deflection for a uniformly distributed load on it such that the maximum bending stress induced is 120 N/mm2. E = 2 x 105N/mm2.

Given data:

180° = π radians



d = 1800mm

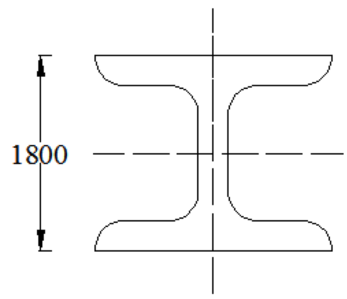
ℓ = 20m = 20000mm

σ_b = 120 N/mm^2

E = 2 x 10^5 N/mm^2

To find:

δmax = ?



Solution:

Moment of Inertia (I)

We know

$\frac{M}{I} = \frac{\sigma_b}{y} = \frac{E}{R}$

$$y = \frac{d}{2} = \frac{1800}{2} = 900\text{mm}$$

$$M = \frac{w\ell^2}{8} = \text{B.M.}$$

$$\sigma_b = \frac{M}{I} \times y = \frac{w\ell^2}{8} \times \frac{900}{I} = 112.5\left(\frac{w\ell^2}{I}\right)$$

$$\sigma_b = 112.5\left(\frac{w\ell^2}{I}\right) = 120 \text{ N/mm}^2$$

$$\left(\frac{w\ell^2}{I}\right) = \frac{120}{112.5} = 1.067$$

$$\delta_{max} = \frac{5\ell^2}{384E}\left(\frac{w\ell^2}{I}\right)$$

$$\delta_{max} = \frac{5 \times (20000)^2}{384 \times 2 \times 10^5} \times (1.067) \text{ mm}$$

$$\delta_{max} = 27.79\text{mm}$$

Result

$\delta_{max} = 27.79$mm

Problem 18

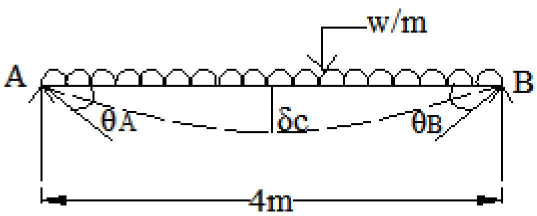
A Simply Supported beam 4m long carries an udl over entire span if the maximum slope is 1° calculate the max deflection also udl.
Take EI = 1 x 10^4 kN.m^2

Given:

ℓ = 4m = 4000mm

$$\theta_A = \theta_B = \frac{\pi}{180} \times 1°$$
$= 17.45 \times 10^{-3}$ radian

EI = 1 x 10^4 kN.m^2



Required:

$\delta_{max} = ?$

Solution:

(i) Safe udl (w)

Simply supported beam with udl over entire span

We know,

Slope $\theta = \dfrac{w\ell^3}{24EI}$

$$w = \frac{\theta \text{ x } 24\text{EI}}{\ell^3}$$

$$= \frac{17.45 \text{ x } 10^{-3} \text{ x } 24 \text{ x } 1 \text{ x } 10^4}{4^3}$$

$= 65.44$ kN/m

Deflection $\delta_c = \dfrac{5w\ell^4}{384EI}$

$$\delta_{max} = \frac{5 \text{ x } 65.43 \text{ x } 4^4}{384 \text{ x } 1 \text{ x } 10^4} = 21.81 \text{ x } 10^{-3} \text{ m}$$

$\delta_{max} = 21.81$mm

Result

w $= 65.44$ kN/m

$\delta_{max} = 21.81$mm

Problem 19

A beam of span 4.5m is Simply Supported at its ends. Calculate the maximum permitted udl, if the maximum slope at the support is restricted to 1° also calculate the max deflection also udl. Take EI = 0.8 x 10^4 kN.m^2

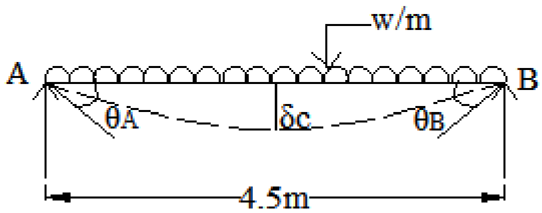
Given

$\ell = 4.5$m $= 4500$mm

$\theta_A = \theta_B = \dfrac{\pi}{180}$ x 1°

$= 17.45 \text{ x } 10^{-3}$ radian

EI $= 0.8 \text{ x } 10^4$ kN.m^2



Required

w = ? , $\delta_{max} = ?$

Solution

For a simply supported beam with udl over entire span

We know,

Max. Slope $\theta_A = \dfrac{w\ell^3}{24EI}$

$$w = \frac{\theta_A \times 24\,EI}{\ell^3}$$

$$= \frac{17.45 \times 10^{-3} \times 24 \times 0.8 \times 10^4}{(4.5)^3}$$

w = 36.77 kN/m

Deflection $\delta_c = \dfrac{5\,w\ell^4}{384\,EI}$

Deflection $\delta_c = \dfrac{5\,w\ell^4}{384\,EI}$

$$\delta_{max} = \frac{5 \times 36.77 \times 4.5^4}{384 \times 0.8 \times 10^4} = 24.54 \times 10^{-3} \text{ m}$$

δ_{max} = 24.54mm

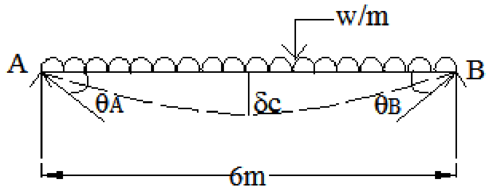
Result

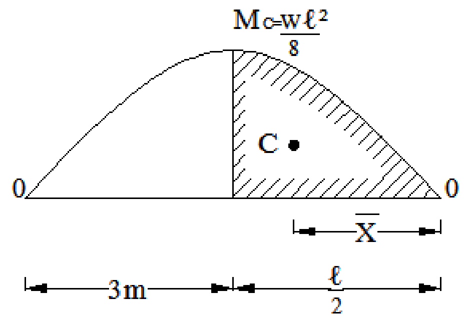
w = 36.77 kN/m

δ_{max} = 24.54 mm

Problem 20

A steel pipe 50mm internal dia. 2.5mm thick is simply supported over a span of 6m, if the deflection is limited to $\dfrac{1}{325}$ of span calculate the rate of loading and maximum slope. Take E = 2 x 10^5N/mm^2. Also find the maximum slope.





Given data:

Internal dia d = 50 mm

Thickness t = 2.5 mm

External dia D = (50 + 2 x 2.5) = 55 mm

Span ℓ = 6m = 6000 mm

$$\delta_{ma} = \frac{1}{325} \times \text{span} = \frac{6}{325}$$

$= 18.46 \times 10^{-3}$m

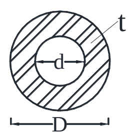
E = 2 x 10^5N/mm^2

Required:

w = ?

Solution:

(i) Moment of inertia (I)



$$I = \frac{\pi}{64}(D^4 - d^4)$$

$$I = \frac{\pi}{64}(55^4 - 50^4)$$

$I = 142.38 \times 10^3\ mm^4$

$EI = 2 \times 10^5 \times 142.39 \times 10^3$

$EI = 28.48 \times 10^9\ N.mm^2 = 28.48\ kN.m^2$

We know,

By Area Moment Method

(ii) Free BMD

BM due to udl = $\frac{w\ell^2}{8} = \frac{w \times 6^2}{8}$ = 4.5w kN.m

Draw BMD

(iii) Area of BMD

Area of Parabola a = $\frac{2}{3}$ x 3 x 4.5 w = 9 w

Centroid $\bar{x} = \frac{5}{8}$ x 3 = 1.875 m from B

(iv) Safe udl (w)

By Mohr's theorem – II

$\delta_{max} = \frac{A\bar{x}}{EI} = \frac{1}{EI}\left(a\bar{x}\right) = 18.46 \times 10^{-3}$

$\delta_{max} = \frac{1}{28.48}$ [(9 w x 1.875)] = 18.46×10^{-3}

w = 31.16×10^{-3} kN/m

(v) Slope (θ)

By Mohr's theorem – I

$\theta_A = \theta_B = \frac{A}{EI} = \frac{1}{EI}[A]$

$\frac{1}{EI}[9w] = \frac{1}{28.48}[9 \times 31.16 \times 10^{-3}]$

$\theta_A = 9.847 \times 10^{-3}$ radians

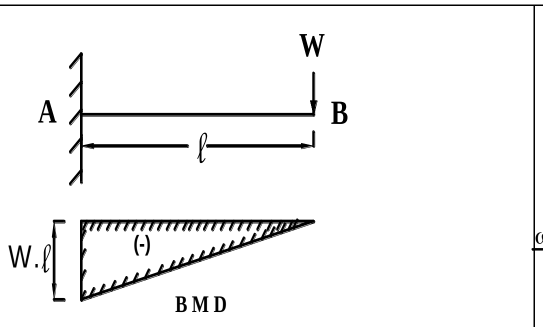
Results

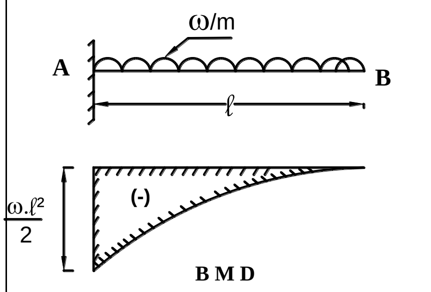
w = 31.16×10^{-3} kN/m

$\theta_A = 9.847 \times 10^{-3}$ radians

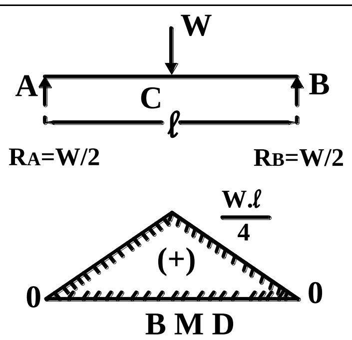
Bending Moment diagram for standard cases of beam

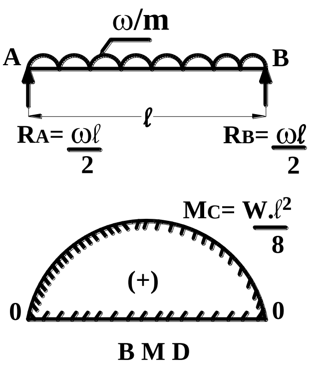
1.





2.





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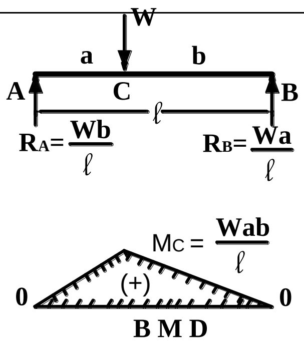
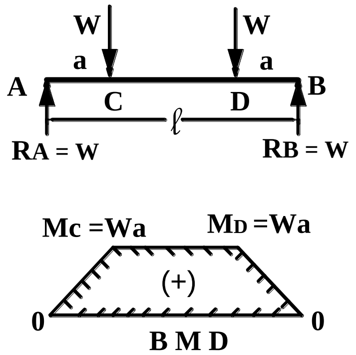


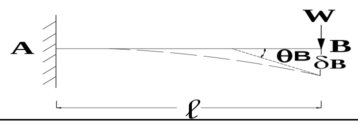
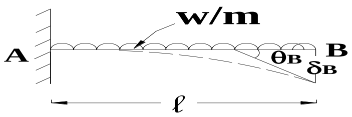
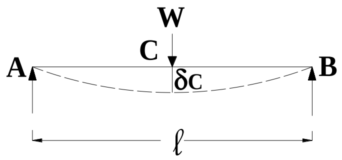
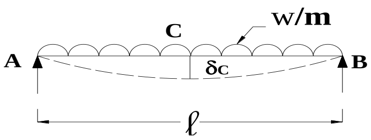


1.1 HIGHLIGHTS

1. Mohr's Theorem – I $\theta = \dfrac{A}{EI}$ = slope

2. Mohr's Theorem – II $y = \dfrac{A\bar{x}}{EI}$ = deflection.

3. Slope and Deflection

Beam	Slope	Deflection	Deflect / Slope Ratio
	$\theta_B = \dfrac{W\ell^2}{2EI}$	$\delta_B = \dfrac{W\ell^3}{3EI}$	$\dfrac{2}{3}\ell$
	$\theta_B = \dfrac{w\ell^3}{6EI}$	$\delta_B = \dfrac{w\ell^4}{8EI}$	$\dfrac{3}{4}\ell$
	$\theta_A = \theta_B = \dfrac{W\ell^2}{16EI}$	$\delta_C = \dfrac{W\ell^3}{48EI}$	$\dfrac{1}{3}\ell$
	$\theta_A = \theta_B = \dfrac{w\ell^3}{24EI}$	$\delta_C = \dfrac{5w\ell^4}{384EI}$	$\dfrac{5}{16}\ell$

1.1 QUESTIONS for Homework

Part I

Each question consist only 2 marks

1. What is deflection of beam at free end for the beam carrying Point load at Free end?
2. What is the slope at support of a simply supported beam carrying point load at mid span?
3. Define: Indeterminate Structure.
4. Define: Slope and Deflection.
5. State the Maximum slope value in a simply supported beam subjected to a point load at mid span.
6. Write the differential equation of flexure.
7. Draw the deflected shapes of any two beams.
8. Write the difference between roller and hinged supports.
9. Define elastic curve?
10. Write the equation of area moment method theorem of deflection

Part II

Three / Five mark Questions

1. State Mohr's area moment theorems.
2. State Mohr's area moment theorems for slope and deflection.
3. State Mohr's Theorems 1 and 2 with respect to the deflected shape of a beam.
4. A cantilever beam of 3 meter length is subjected to a point load of 30 kN at its free end. Find the deflection at the free end, using formula. if EI = 90 x 10^{12} N.mm2.
5. Explain slope and deflection

Part III

Ten mark Questions

1. A steel pipe 50mm internal diameter and 2mm wall thickness is simply supported on a span of 6m. If the deflection is limited to 1/325 of the span, calculate the rate of loading on the beam. Also calculate the maximum slope at the supports. Take E = 2 x 10^5 N/mm^2.
2. A cantilever beam 120 mm wide and 200 mm deep is 3 m long. What udl should the beam can carry to produce a deflection of 8 mm at the free end. Take E = 210 GN/mm^2.
3. A cantilever of 5 meter span carries an u.d.l. of intensity 20kN/m over a length of 3m from its fixed end and two point loads of 40kN and 30kN at 3m and 5m from the fixed end respectively. Determine the maximum slope and deflection at the free end using Mohr's Theorems if EI = 47.05 x 10^3kNm^2.
4. A cantilever beam of span 4m is subjected to an UDL of 20kN/m over the entire length and a point load of 30 kN is acting at free end. Calculate the slope and Take E = 2 x 10^5 N/mm^2 and I = 8 x 10^7mm^4.
5. A cantilever beam of 1 m long is of rectangular section of width 40 mm and depth 60 mm. calculate the maximum udl that can be allowed over the entire length of the beam without exceeding a deflection of 3.5 mm at the free end. Also calculate the maximum slope at the free end. Take E = 7 x 10^4 N/mm^2.
6. A beam of span 4.5m is simply supported at its ends. Calculate the maximum permitted udl if the maximum slope at the support is restricted to 1°. Also calculate the maximum deflection. Take EI = 0.80 x 10^4 kNm^2.
7. A simply supported beam of span 4m carries an UDL of 10 kN/m over the full length and a central point load of 20kN . Determine the maximum slope and maximum deflection by area moment method. Take E=2x10^5N/mm^2 and I=4x10^8mm^4 and I = 8x10^7mm^4
8. A simply supported beam of span 8m carries an UDL of 18 kN/m throughout its length and a concentrated load of 60kN at the centre. EI = 2x10^5 KN.m^2 Determine the maximum values of slope and deflection in the beam, using Mohr's theorem. (Formula shall not be used).

1.2 PROPPED CANTILEVERS

Statically determinate and indeterminate Structures- Stable and Unstable Structures-Examples- Degree of Indeterminacy-Concept of Analysis of Indeterminate beams - Definition of Prop –Types of Props- Prop reaction from deflection consideration – Drawing SF and BM diagrams by area moment method for UDL throughout the span, central and non-central concentrated loads – Propped cantilever with overhang – Point of Contra flexure.

1.2.1 Static Equilibrium Equations

According to the principle of statics, any structural member should satisfy the following equilibrium conditions.

1. Algebric sum of all vertical forces should be equal to zero.

   ie $\Sigma v = 0$

   Sum of upward vertical forces = Sum of down ward vertical forces.

   $$(+)\uparrow v = \underset{V}{\downarrow}(-)$$

2. Algebric sum of all horizontal forces should be equal to zero.

   ie $\Sigma H = 0$

   Sum of forces towards right side = Sum of forces towards side

   $$H \underset{(+)}{\rightarrow} = \underset{(-)}{\leftarrow} H$$

3. Algebric sum of moments of all forces should be equal to zero.

   ie $\Sigma M = 0$

   $$(+) \circlearrowleft = \circlearrowright (-)$$

   Sum of anticlockwise moments = Sum of clockwise moments.

Determinate and indeterminate beams

Based on the static equilibrium equations the beams are classified as follows.

1. Statically determinate beams.
2. Statically indeterminate beams.

2.1.5. Statically determinate beams

When the reaction components of a beam **can be** analysed by using static equilibrium equations ($\Sigma_V = 0$, $\Sigma_H = 0$, $\Sigma_M = 0$) only, it is called as statically determinate beam. In which the degree of indeterminacy is equal to zero.

The examples for statically determinate beams as given below

i) Cantilever beam
ii) Simply supported beam
iii) Overhanging beam

1.2.1. (b) Statically indeterminate beam

When the reaction components of a beam **cannot be** analysed by using static equilibrium equations ($\Sigma_V = 0$, $\Sigma_H = 0$, $\Sigma_M = 0$) only. The beam is called as statically indeterminate beam. In which the degree of indeterminacy is not equal to zero. (One or more than one).

The examples for statically indeterminate beams are given below

i) Propped cantilever beam ii) Fixed beam iii) Continuous beam

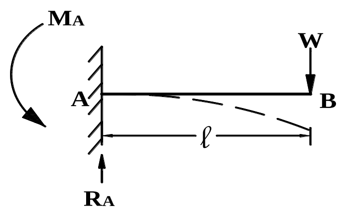
1.2.1 (c) Degree of Indeterminacy

The difference between No. of unknown reaction components and No. of known equilibrium equation is called Degree of Indeterminacy or degree of redundancy.

$$\text{Degree of indeterminacy} = \begin{pmatrix}\text{No. of unknown} \\ \text{reaction components}\end{pmatrix} - \begin{pmatrix}\text{No. of known using static} \\ \text{equilibirum equations}\end{pmatrix}$$

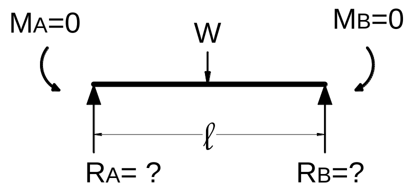
1.2.1. (c) Example

1. Cantilever beam



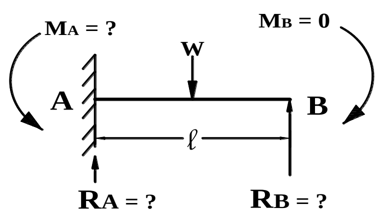
Unknown reaction components M_A & R_A = 2
Know static equilibrium equation ($\Sigma v = 0$; $\Sigma_M = 0$) = 2
$\therefore$ D.I = (2 – 2) = 0
It is statically determinate beam.

2. Simply Supported beam



Unknown reaction components R_A & R_B = 2
Known static equilibrium equation ($\Sigma_M = 0$; $\Sigma_V = 0$) = 2
$\therefore$ D.I = (2 – 2) = 0
It is statically determinate beam.

3. Propped cantilever beam

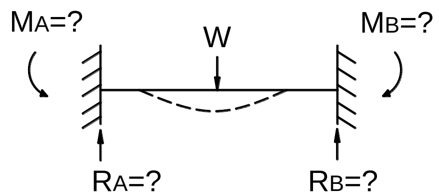


Unknown reaction components R_A, R_B & M_A = 3
Known static equilibrium equation ($\Sigma v = 0$; $\Sigma_M = 0$) = 2
$\therefore$ D.I = (3 – 2) = 1
It is statically indeterminate beam.

4. Fixed beam

Unknown reaction components

(R_A, R_B, M_A, M_B) = 4 Nos.



Known static equilibrium equations

($\Sigma v = 0$, $\Sigma M = 0$) = 2

$\therefore$ D.I. = (4-2) = 2 > 0

It is statically indeterminate beam.

5. Continuous beam

Unknown reaction components

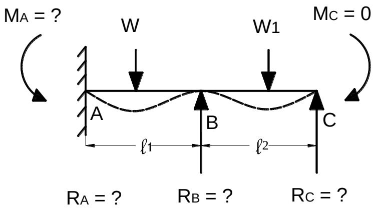
(R_A, R_B, Rc, M_A, M_B) = 5 Nos.

Known static equilibrium equations

($\Sigma v = 0$, $\Sigma M = 0$) = 2

$\therefore$ D.I. = (5 – 2) = 3

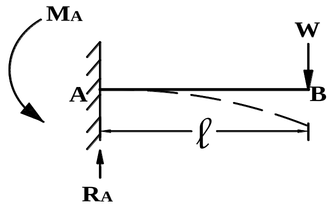
It is statically indeterminate beam.



1.2.3. Method of analysis of indeterminate beam

The following are the various methods of analysis of indeterminate beams.

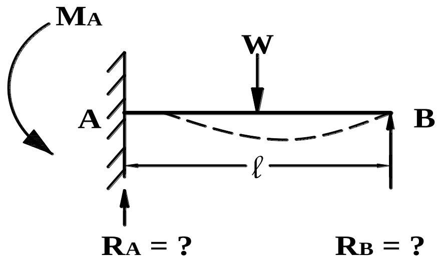
1. Area moment method.
2. Theorem of three moment method.
3. Moment distribution method.
4. Strain energy method.
5. Column analogy method
6. Slope deflection method.

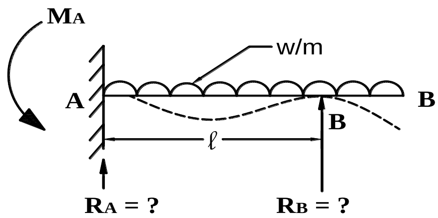


1.2.4 Propped cantilever beam

Definition

When a cantilever beam is supported by vertical post at free end (or) near the free end is called Propped cantilever beam. It is statically determinate beam and its degree of indeterminacy is 1.





Advantages of Propped cantilever beam

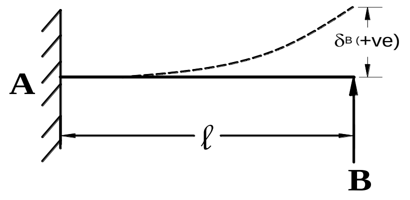
By providing propped at free end (or) nearby free end in a cantilever beam, the following are the advantages.

1. Deflection at prop. is zero.
2. More shift and stable.
3. It can carry more load than cantilever beam.
4. Deflection is reduced.
5. Maximum +ve bending moment will be induced near the middle span.
6. Value of hogging moment at support is reduced.

1.2.5 Type of Prop.

The following are the various types of Prop.

1. Rigid Prop.
2. Sinking Prop.
3. Elastic Prop.



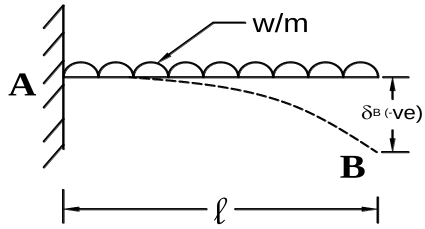
Rigid Prop

i) At Rigid prop. the upward deflection is equal to downward deflection.

ii) There is no change in length of Prop.

i.e. Upward deflection = Downward deflection.

$$\delta_B \text{ (+)ve} = \delta_B \text{ (-)ve}$$



Sinking Prop.

At sinking Prop. a part of the deflection destroys by the load.

i.e. Upward deflection at Prop. End } is not equal to { Down load deflection due to load.

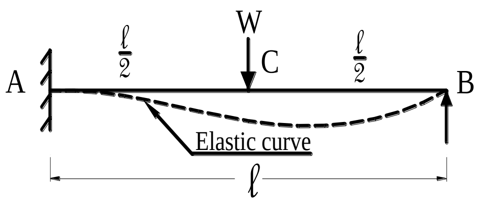
$$\delta_B \text{ (+)ve} \neq \delta_B \text{ (-)ve}$$

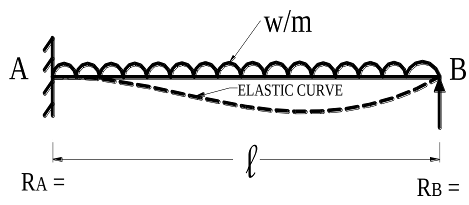
Elastic Prop.

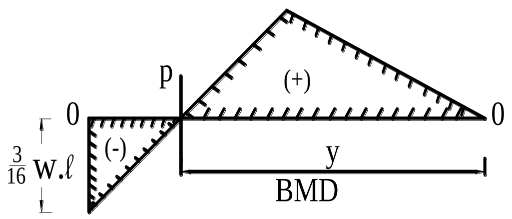
At elastic Prop. a part of deflection destroys and also changes in length due to reaction in the Prop. because the Prop. material is elastic.

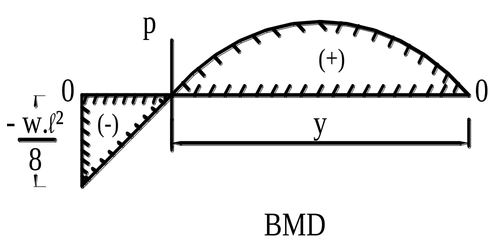
2.3.6 Point of contra flexure

The point where the BMD changes its sign from (+)ve to (–)ve and vice versa as shown in fig. is called point of contra flexure (or) point of inflexion. The B.M at this point is zero.









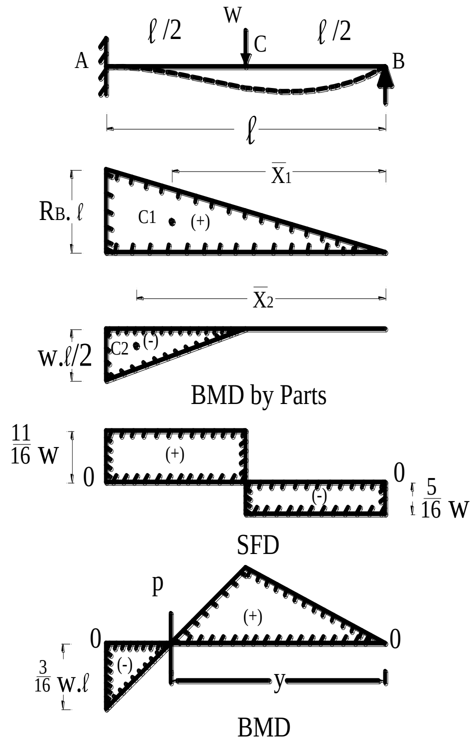
AREA MOMENT METHOD

Book Work – 1

1.2.6 Propped cantilever beam with central point load (W)

Solution

Consider a cantilever beam propped at free end and loaded as shown in fig.



Let R_B = Prop. Reaction

BM at a due to Prop. Reaction = $R_B.\ell$

BM at A due to load = w. $\frac{\ell}{2}$

Draw BMD by parts.

$A_1 = \frac{1}{2}$ x b x h

$A_1 = \frac{1}{2} \text{ x } \ell \text{ x } R_B\, \ell = \frac{R_B\, \ell^2}{2}$

$\overline{x_1} = \frac{2}{3}\, \ell$

$A_2 = (-)\, \frac{1}{2} \text{ x } \frac{l}{2} \text{ x } \frac{W\ell}{2} = \frac{(-)W\ell^2}{8}$

$\overline{x_2} = \left(\frac{\ell}{2}\right) = \frac{5\ell}{6}$

Prop. Reaction

By Mohr's Theorem – II

$\delta\text{max} = \frac{A\overline{x}}{EI} = 0$

$A\overline{x} = 0;\ A_1\overline{x}_1 + A_2\overline{x}_2 = 0$

$$\left(\frac{R_B\, \ell^2}{2}\right)\frac{2}{3}l - \left(\frac{W\ell^2}{8}\right)\left(\frac{5\ell}{6}\right) = 0$$

$$\frac{R_B\, \ell^3}{3} = \frac{5\, W\ell^3}{48}$$

$$R_B = \frac{3 \text{x} 5W}{48} = \frac{5}{16} \text{ W}$$

$$\boxed{R_B = \frac{5}{16} \text{ W}}$$

$R_A = W - R_B = W - \frac{5}{10} W = \frac{11}{16}$ W

Draw SFD as shown in fig.1.37

Bending Moment

$M_B = 0$

$M_C = \frac{5}{16}\ W \text{ x } \frac{l}{2}\ = \frac{5W\ell}{32}$

$M_D = \left(\frac{5}{16}W\,x\,\ell\right) - \left(W\,\frac{l}{2}\right) = W\,\ell\left(\frac{5-8}{16}\right)$

$\boxed{M_A = \frac{-3}{16}W\,\ell}$

Draw BMD

Point of Contra Flexure (P) :

Point of contra flexure will occur at "y" distance from B.

$M_y = 0$

$M_y = R_B \text{ x } y - w\,(y - \frac{\ell}{2}) = 0$

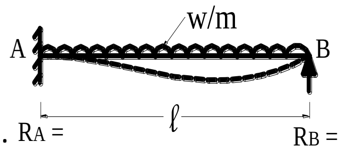
$\frac{5}{16}\ .\ w\ .\ y - w.\ y + \frac{w.\ell}{2}\ = 0$

$\frac{11}{16}.y = \frac{\ell}{2}$

$\boxed{y = \frac{8}{11}\ell}$ from B

<u>Book Work – 2</u>

1.2.7 Propped Cantilever beam with udl (w)

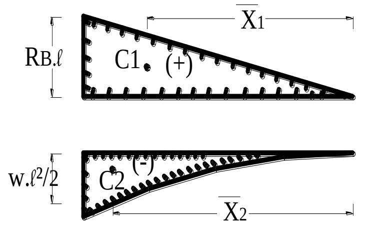


Solution

Consider a propped cantilever beam loaded as shown in fig.

Let R_B = Prop. Reaction

Draw BMD by parts.



BMD by Parts

Prop. Reaction

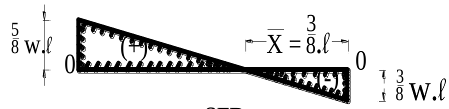
$A_1 \quad = \frac{1}{2} \text{ x b x h}$

$A_1 \quad = \frac{1}{2} \text{ x } \ell \text{ x } R_B\ \ell = \frac{R_B\,\ell^2}{2}$

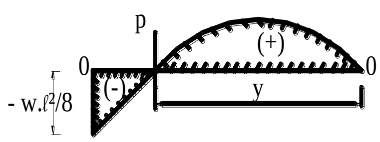
$\overline{x_1} \quad = \frac{2}{3}\ \ell$

$A_2 \quad = \frac{1}{3} \text{ x } \ell \text{ x } \frac{w\ell}{2} = \frac{w\ell^3}{6}$

$\overline{x_2} \quad = \frac{3}{4}\ell$



SFD



BMD

By Mohr's Theorem – II

δmax = $\dfrac{A\bar{x}}{EI}$ = 0

A $\bar{x}$ = 0

$\left(\dfrac{R_B\,\ell^2}{2}\right)\left(\dfrac{2}{3}\ell\right) - \left(\dfrac{w\ell^2}{6_2}\right)\left(\dfrac{3}{4}\ell\right) = 0$

$\dfrac{R_B\,\ell^3}{3} = \dfrac{w\ell^4}{8}$

$\boxed{R_B = \dfrac{3}{8}\,w\,\ell}$

$\therefore$ $R_A = w \times \ell - R_B = w\ell - \dfrac{3}{8}\,w\ell$

$\boxed{R_A = \dfrac{5}{8}\,w\,\ell}$

Draw SFD

Max. BM will occur at 'x' distance from B

$\dfrac{x}{\frac{3}{8}w\ell} = \dfrac{(\ell - x)}{\frac{5}{8}w\ell}$

6x = 3 ℓ – 3 x

8x = 3 ℓ

$\boxed{x = \dfrac{3}{8}\ell}$

Bending Moment

M_B = 0

M_x = $R_B .x - w.\,x.\,\dfrac{x}{2}$

= $\left(\dfrac{3}{8}w\ell\right)\left(\dfrac{3}{8}\ell\right) - w\left(\dfrac{3}{8}\ell\right) \text{x} \dfrac{1}{2} \text{x} \left(\dfrac{3}{8}\ell\right)$

M_x = $\dfrac{9\,w\ell^2}{6} - \dfrac{9\,w\ell^2}{128} = \dfrac{18\,w\ell^2 - 9\,w\ell^2}{128} = \dfrac{9\,w\ell^2}{128}$

M_x = $\dfrac{9\,w\ell^2}{128}$

M_A = $R_B \times \ell - w \times \ell \times \dfrac{l}{2}$

= $\dfrac{3}{8}\,w\ell \times \ell - w \times \dfrac{\ell^2}{2} = W\ell^2 \left(\dfrac{3-4}{8}\right)$

$M_A = -\dfrac{w\ell^2}{8}$

Draw BMD

Point of Contra Flexure (P) :

Point of contra flexure will occur at "y" distance from B.

$M_y = 0$

$$My = R_B \times y - w.\, y\, .\frac{y}{2} = 0$$

$$\frac{3}{8}\, .\, w\,\ell .\, y - w.\, \frac{y^2}{2} = 0$$

$$w.\, \frac{y^2}{2} = \frac{3}{8}\, l.y$$

$$\boxed{y = \frac{3}{4}\ell} \quad \text{from B}$$

<u>Book Work – 3</u>

1.2.8 Propped cantilever beam with Non-central point load (W)

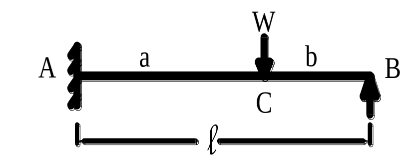
Solution

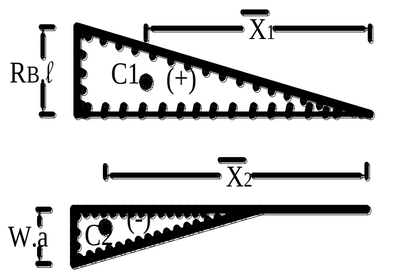
Consider a cantilever beam propped at free end and loaded as shown in fig.

Let R_B = Prop. Reaction at B

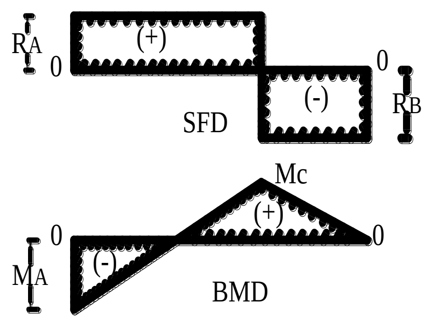
BM at a due to Prop. Reaction = $R_B.\ell$

BM at A due to load = w. a





BMD by Parts



i) Area

$$a_1 = \frac{1}{2} \times \ell \times R_B.\ell = \frac{R_B.\ell^2}{2}$$

$$\overline{x_1} = \frac{2}{3} \times \ell \qquad = 4m$$

$$a_2 = \frac{1}{2} \times a \times W.a = \frac{W.a^2}{2}$$

$$\overline{x_2} = [\frac{2}{3} \times a] + b$$

ii) Prop. Reaction (R_B)

By Mohr's Theorem - II

$$\delta max = \frac{A\overline{x}}{EI} = 0, \; A\overline{x} = 0$$

Upward deflection at B due to R_B = δ_B (+)

$$\delta_B (+) = \frac{A\overline{x}}{EI} = \frac{1}{EI} A_1 \overline{x}$$

δ_B (+) = $\frac{1}{EI}\left[\frac{R_B.\ell^2}{2} x \frac{2}{3}\ell\right]$ = $\left[\frac{R_B.\ell^3}{3EI}\right]$-------------- (1)

Downward deflection at B due to load W = δ_B (–)

δ_B (–) = $\frac{A_2 \overline{x_2}}{EI}$ = $\frac{1}{EI}\left[\frac{w.a^2}{2} x\left(b+\frac{2}{3}b\right)\right]$

δ_B (–) = $\left[\frac{w.a^2}{6EI}(3\ell - a)\right]$ -------------- (2)

at Rigid Prop. Upward deflection = Downward deflection

δ_B (+) = δ_B (–)

$\left[\frac{R_B.\ell^3}{3EI}\right]$ = $\left[\frac{w.a^2}{6EI}(3\ell - a)\right]$

R_B = $\left[\frac{w.a^2}{2.\ell^3}(3\ell - a)\right]$

R_A = Total load – R_B

= $\left[W - \frac{w.a^2}{2.\ell^3}(3\ell - a)\right]$

R_A = $\left[\frac{w.b}{2.\ell^3}(3\ell^2 - b^2)\right]$

Draw SFD

Bending Moment

$M_B = 0$

$M_C = R_B$ x b = $\left[\frac{w.a^2}{2.\ell^3}(3\ell - a) x b\right]$ = $\left[\frac{w.a.b^2}{2.\ell^3}(3\ell - a)\right]$

$M_A = R_B$ x ℓ – w.a = $\left[\frac{w.a^2}{2.\ell^3}(3\ell - a) - (W.a)\right]$ = – $\left[\frac{w.b}{2.\ell^2}(\ell^2 - b^2)\right]$

Draw BMD

Problem 1

A propped cantilever beam 4m long carries a central point load of 20 kN. Determine the prop. reaction and draw SFD and BMD.

Given data:

Span ℓ = 4 m

Load W = 20 KN

Solution:

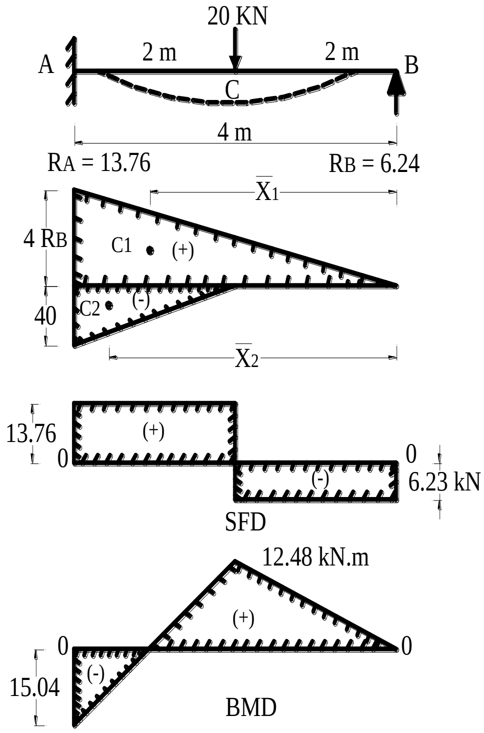
i) Bending moment

R_B = Prop. Reaction

BM at A due to R_B = R_B x 4

BM at A due to load = 20x2 = 40 KN.m

Draw BMD



ii) Area of Bending Moment Diagram

$A_1 = \frac{1}{2} \text{xbxh}$

$A_1 = \frac{1}{2} \times \ell \times h$

$= \frac{1}{2} \times 4 \times 4 R_B$

$\boxed{A_1 = 8 R_B}$

$\overline{x_1} = \frac{2}{3} \times 4 = 2.67$ m

$A_2 = \frac{1}{2} \times b \times h$

$A_2 = \frac{1}{2} \times 2 \times 40 = -40$ kNm^2

$\overline{x_2} = 2 + \frac{2}{3} \times 2 = +3.33$ m

iii) Prop. Reaction (R_B)

By Mohr's Theorem – II

$\delta\text{max} = \frac{A\overline{x}}{EI} = 0$

$A\overline{x} = 0$

$A_1 \overline{x_1} + A_2 \overline{x_2} = 0$

$(8 R_B \times 2.67) - 40 \times 3.33 = 0$

$8R_B \times 2.67 = 40 \times 3.33$

$\boxed{R_B = 6.24KN}$

$R_A = 20 - 6.23$

$\boxed{R_A = 13.76KN}$

Upward deflection = Downward deflection

$\delta_B (+) = \delta_B (-)$

$\frac{a_1 \overline{x_1}}{EI} = \frac{a_2 \overline{x_2}}{EI}$

$a_1 \overline{x_1} = a_2 \overline{x_2}$

8 RB x 2.67 = 40x3.33

R_B = 6.24 KN

Reaction = Total load – R_B

$R_A = 20 - 6.24$

$R_A = 13.76$ kN

Bending Moment

$M_B = 0$

$M_C = (6.24 \times 2) = 12.48$KN.m

M_A = (6.24 x 4) – (20 x 2) = -15.04KN.m

Draw BMD

Point of Contra Flexure (y) :

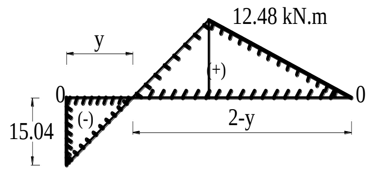
Point of contra flexure will occur at "y" distance from B. (Similar triangles)

$$\frac{y}{15.04} = \frac{2-y}{12.48}$$

12.48 y = 15.04 (2 – y) = 0

27.52 y = 30.08

y = $\frac{30.08}{27.52}$ = 1.09



$\boxed{y = 1.09}$ m from A

Problem 2

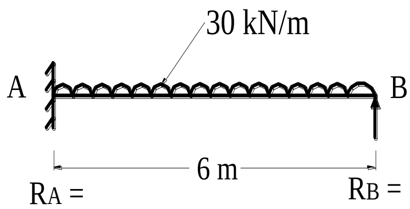
A Propped cantilever beam is 6m long. It carries an udl of 30 KN/m over its entire span. Determine the Prop. reaction and Draw SFD and BMD.

Solution

Span ℓ = 6 m

udl w = 30 KN/m

let R_B = Prop. reaction



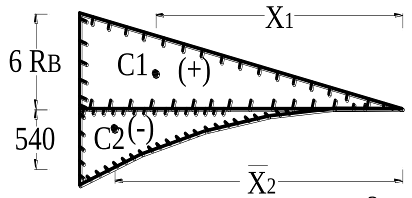
i) Bending moment by parts

R_B = Prop. Reaction

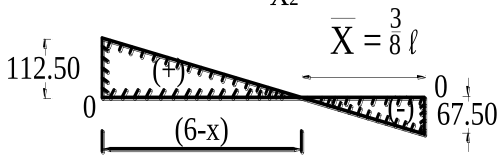
BM at A due to R_B = R_B x ℓ = 6 R_B

BM at A due to udl = $\frac{30\,X\,6^2}{2}$

= – 540 KN.m



Draw BMD



SFD

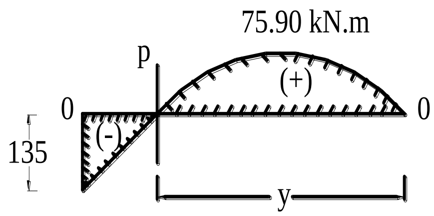
ii) Area

a_1 = area of BMD due to R_B

$A_1 = \frac{1}{2}$ x 6 x 6 R_B = 18 R_B

Centroid

$\overline{x_1} = \frac{2}{3} x 6$ = 4 m



BMD

a_2 = area of BMD due to udl

$A_2 = \frac{1}{3}$ x 6 x 540 = 1080 KN.m^2

Centroid of BMD from B

$\overline{x_2} = \left(\frac{3}{4} x 6\right) = 4.5\text{m}$

ii) Prop. Reaction

By Mohr's Theorem – II

Upward deflection due to Prop. reaction = Downward deflection due to load

$$\frac{A_1 \overline{x_1}}{EI} = \frac{A_2 \overline{x_2}}{EI}$$

$$A_1 \overline{x_1} = A_2 \overline{x_2}$$

18 RB x 4 = 1080 x 4.5

R_B = 67.50 KN

Reaction at A = Total load – R_B

R_A = (30 x 6) – 67.50 = 112.50 kN

Draw SFD

Maximum BM will occur at 'x' distance from B

$$\frac{x}{67.50} = \frac{6-x}{112.50}$$

112.50 x – 67.50 (6 – x) = 405 – 67.50 x

x = 2.25 m from B

Bending Moment

M_B = 0

Mx = R_B x 2.25 – w.x. $\frac{x}{2}$ = 67.50 x 2.25 – 30 x $\frac{2.25^2}{2}$ = 75.94 kN.m

M_A = R_B x ℓ – $\frac{w.\ell^2}{2}$ = (67.5 x 6) – (30x$\frac{6^2}{2}$) = – 135 KN.m

Draw BMD

Point of Contra Flexure (P) :

Point of contra flexure will occur at "y" distance from B.

M_y = 0

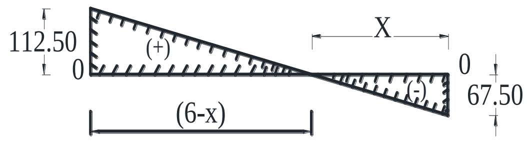
My = R_B x y – w. y . $\frac{y}{2}$ = 0

67.50 x y – 15 y^2= 0

y^2 - 4.5 = 0

y^2 = 4.5

$\boxed{y = 2.12 \text{ m}}$ from B

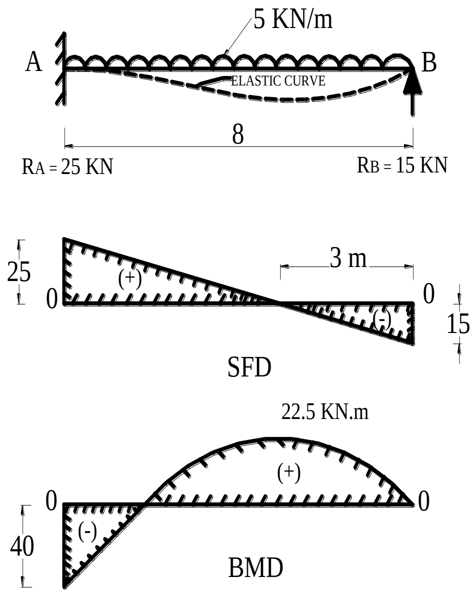


Problem 3

A propped cantilever beam 8m span carries an udl of 5 kN/m over its entire length. Determine the prop. reaction also draw SFD and BMD.

Solution

i) Prop. reaction



we know, $R_B = \frac{3}{8} w\ell$

$R_B = \frac{3}{8}$ x 5 x 8 = 15 kN

R_A = 5 x 8 – 15 = 25kN

Draw SFD

Max. (+ve) BM will occur at 'x' distance from Prop.

$$\frac{x}{15} = \frac{8-x}{25}$$

$25x = 15(8 - x)$

$25x = 120 - 15x$

$40x = 120$

$x = \frac{120}{40} = 3\text{m}$

Bending Moment

$M_B = 0$

$M_x = (15 \text{ x } 3) - \left(5 \text{ x } 3 \text{ x} \frac{3}{2}\right) = 22.5 \text{ kN.m}$

$M_A = 15 \text{ x } 8 - \left(5 \text{ x } 3 \text{ x} \frac{8}{2}\right) = -40 \text{ kN.m}$

Check

$M_A = \frac{-w\ell^2}{8} = \frac{5x8^2}{8} = -40 \text{ kN.m}$

Problem 4

A propped cantilever beam 6m span carries a point load of 30 kN at 2m from propped end. Find prop. reaction draw SFD and BMD.

Given data:

Span ℓ = 6 m

Load W = 30 KN

a = 4m ; b = 2m

Solution:

i) Bending moment by parts

BM due to Prop. Reaction = R_B x ℓ = 6 R_B

BM at A due to load = 30x4 = 120 KN.m

ii) Area of BMD

$A_1 = \frac{1}{2}$ x b x h

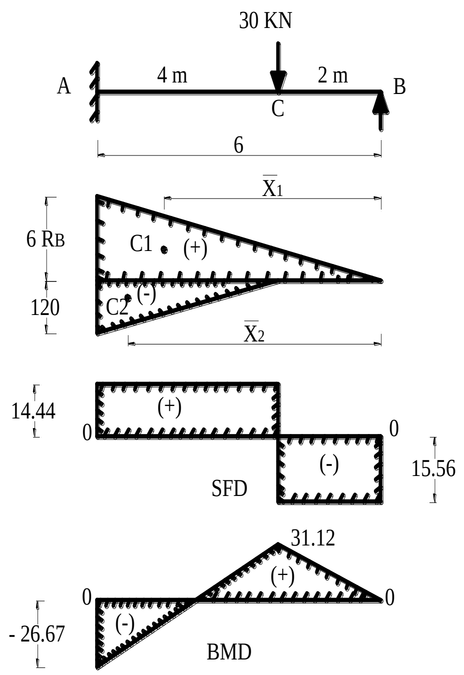
$A_1 = \frac{1}{2}$ x 6 x 6 R_B = 18 R_B

$\overline{x_1} = \frac{2}{3}$ x 6 = 4m

$A_2 = \frac{1}{2}$ xbxh

$A_2 = \frac{1}{2}$ x 4 x 120 = – 240 m^2

$\overline{x_2}$ = 2 + $\frac{2}{3}$ x 4 = 4.67 m



ii) Prop. Reaction (R_B)

By Mohr's Theorem - II

$\delta_{B/A} = \frac{A\bar{x}}{EI} = 0, \; A\bar{x} = 0$

(18 R_B x 4) – (240 x 4.67) = 0

R_B = 15.56 kN

R_A = 30 – 15.56 = 14.44 kN

Draw SFD

Check

(or) we know

$R_B = \frac{Wa^2}{2\ell^3}$ (3ℓ - a)

$R_B = \frac{30 \text{ x } 4^2}{2\ell^3}$ (3 x 6 – 4) = 15.56 kN

R_A = 30 – 15.56 = 14.44 kN

Bending Moment

M_B = 0

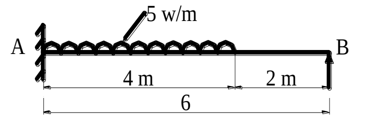
M_C = 15.56 x 2 = 31.12 kN.m

M_A = 15.56 x 6 – 30 x 4 = - 26.67 kN.m

Draw BMD

Problem 5

A beam of length 6m is fixed at one end and supported by a rigid Prop. at the other end at the same level. It carries an udl of 5 KN/m for a length of 4m from fixed end. Determine the Prop. reaction and Draw SFD and BMD.



Given data:

Span ℓ = 6 m

udl w = 5 KN/m

ℓ1 = 4 m

let R_B = Prop. reaction

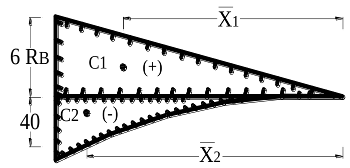
Solution

i) Bending moment by parts

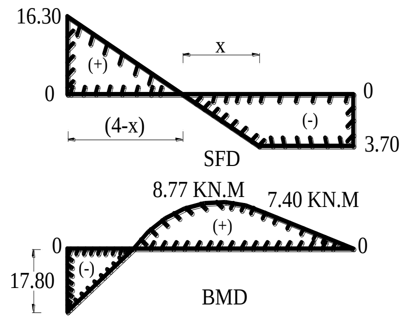
R_B = Prop. Reaction

BM at A due to R_B x ℓ = 4 R_B

BM at A due to udl = $\frac{5 \text{ x } 4^2}{2}$ = – 40 KN.m



Draw BMD by parts



ii) Area of BMD

A_1 = area of BMD due to R_B

$A_1 = \frac{1}{2}$ x 6 x 6 R_B = 18 R_B

Centroid $\overline{x_1} = \frac{2}{3} x 6$ = 4 m from B

A_2 = area of BMD due to udl

$A_2 = \frac{1}{3}$ x 4 x 40 = 53.33 KN.m

$\overline{x_2} = \left(\frac{3}{4} x 4\right) + 2$ = 5m from B

ii) Prop. Reaction (R_B)

By Mohr's Theorem – II

Upward deflection due to Prop. reaction = Downward deflection due to load

$$\frac{A_1 \overline{x_1}}{EI} = \frac{A_2 \overline{x_2}}{EI}$$

18 R_B x 4 = 53.33 x 5

R_B = 3.70 KN

Reaction at A = Total load – R_B

R_A = (5 x 4) – 3.70 = 16.30 kN

Draw SFD

Maximum (+ve) BM will occur at 'x' distance from C

$$\frac{x}{3.70} = \frac{4-x}{16.30}$$

$16.30\ x - 3.70\ (4 - x) = 14.80 - 3.70\ x$

$x = \dfrac{14.80}{20} = 0.74$ m from C

i.e. (2 + 0.74) = 2.74 m from B

Bending Moment

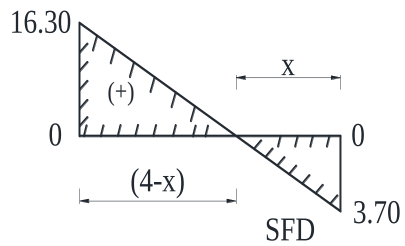
$M_B = 0$

$M_C = R_B \times 2 = 3.7 \times 2 = 7.40$ kN.m

$Mx = R_B \times 2.74 - w.x.\dfrac{x}{2} = 3.70 \times 2.74 - 5 \times \dfrac{0.74^2}{2} = 8.77$ kN.m

$M_A = R_B \times \ell - \dfrac{w.\ell_1^2}{2} = 3.7 \times 6 - \dfrac{5x4^2}{2} = -17.80$ KN.m

Draw BMD



Problem 6

A cantilever beam is supported by vertical post at free end 4m long and it carries a point load of 20 kN at centre and udl of 5 kN/m over entire length. Draw SFD and BMD.

Solution

Span ℓ = 4 m

Load W = 20 KN

udl w = 5 KN/m

i) Bending moment by parts

R_B = Prop. Reaction

BM at A due to R_B = 4 R_B

BM at A due to udl = $\dfrac{5x\ 4^2}{2}$ = 40 KN.m

BM at A due to load = 20x2 = 40 KN.m

Draw BMD by parts

ii) Area of BMD

$A_1 = \dfrac{1}{2} \times b \times h$

$A_1 \quad = \frac{1}{2} \times 4 \times 4\, R_B = 8\, R_B$

$\overline{x_1} \quad = \frac{2}{3} \times 4 \qquad = 2.67$ m

$A_2 \quad = \frac{1}{3} \times b \times h$

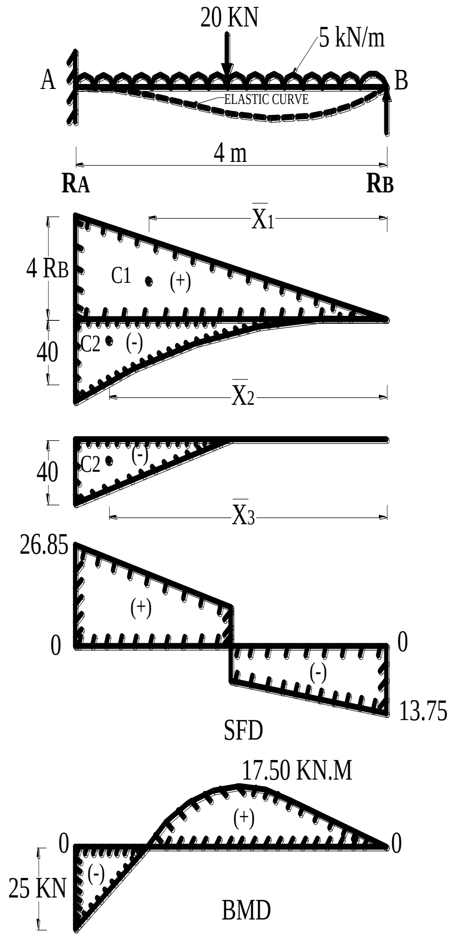
$A_2 \quad = \frac{1}{3} \times 4 \times 40 \qquad = 53.33$

$\overline{x_2} \quad = \left(\frac{3}{4} x 4\right) \qquad = 3$m

$A_3 \quad = \frac{1}{2} \times b \times h$

$A_3 \quad = \frac{1}{2} \times 2 \times 40 \qquad = 40$

$\overline{x_3} \quad = 2 + \frac{2}{3} \times 2 \qquad = 3.33$ m



ii) Prop. Reaction (R_B)

By Mohr's Theorem – II

$\delta_{B/A} = \frac{A\bar{x}}{EI} = 0$

$A\bar{x} = 0$

$\left.\begin{array}{l}\text{Upward}\\ \text{deflection}\end{array}\right\} = \left\{\begin{array}{l}\text{Downward}\\ \text{deflection}\end{array}\right.$

$A_1 \overline{x_1} = [A_2 \overline{x_2} + A_3 \overline{x_3}]$

$(8\, R_B \times 2.67) = [(53.33 \times 3) + (40 \times 3.33)]$

$21.36\, R_B = 293.19$

$R_B \quad = \frac{293.19}{21.36} = 13.75$ kN

$R_A \quad = 20 + (5 \times 4) – 13.75 = 26.25$ kN

(or) we know,

$R_B = \left(\frac{5}{16} W + \frac{3}{8} \omega\ell\right)$

$R_B = \left(\frac{5}{16} \text{x } 20 + \frac{3}{8} \text{x } 5\text{x } 4\right)$

$R_B = 13.75$ kN

$R_A = 20 + 5 \times 4 – 13.75 = 26.25$ kN

Draw SFD

Bending Moment

$M_B = 0$

$M_C = (13.75 \times 2) - (5 \times 2)\left(\frac{2}{2}\right) = 17.50$ kN.m

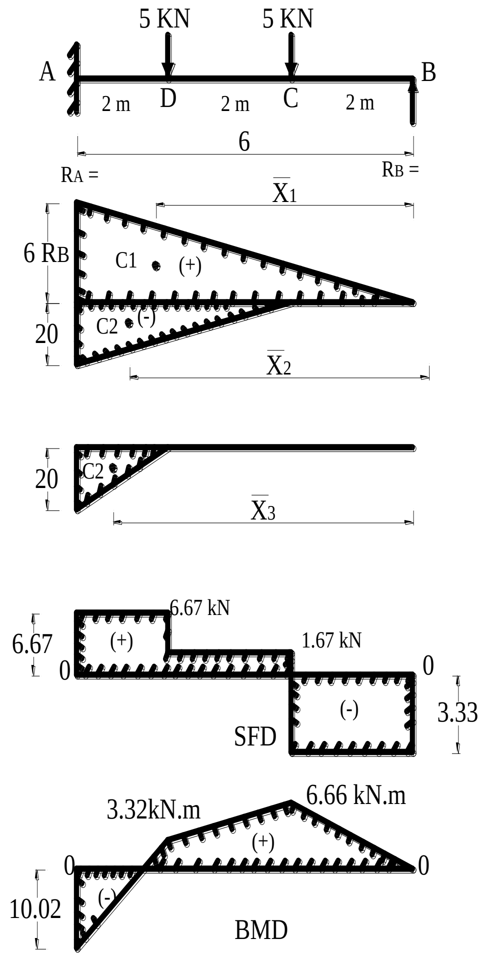
$M_A = 13.75 \times 4 - (20 \times 2) - \left(5 \times 4 \times 4/2\right) = -25$ KN.m

$M_A = -25$ kN.m

Draw BMD

Problem 7

A beam of length 6m is Propped at one end. It carries two point loads of 5 KN each at 2m and 4m from left supports. Determine the Prop. reaction by area moment method and Draw SFD and BMD.



Solution

Span ℓ = 6 m

$W_1 = W_2$ = 5 KN

let R_B = Prop. reaction at B

R_A = Prop. Reaction at A

i) Bending moment by parts

R_B = Prop. Reaction = $R_B \times l = 6R_B$

BM at A due to $W_1 = R_B \times \ell$ (5x4) = 20 KN.m

BM at A due to W_2 = 5 x 2 = 10 KN.m

Draw BMD by parts

ii) Area

A_1 = area of BMD due to R_B

$A_1 = \frac{1}{2} \times 6 \times 6 R_B = 18 R_B$

Centroid $\overline{x_1} = \frac{2}{3} x 6 = 4$ m

A_2 = area of BMD due to udl

$A_2 = \frac{1}{2} \times 4 \times 20 = 40$

Centroid of BMD from B

$\overline{x_2} = 2 + \left(\frac{2}{3} x 4\right) = 4.67$m

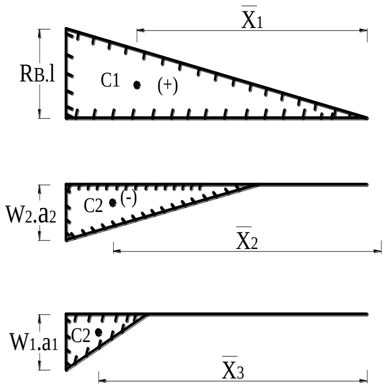
$A_3 = \frac{1}{2} \times 2 \times 10 = 10$

$\overline{x_3} = 4 + \frac{2}{3} \times 2 = 5.33$ m

ii) Prop. Reaction (R_B)
By Mohr's Theorem – II
Upward deflection due to Prop. reaction = Downward deflection due to load

$$\frac{a_1 \overline{x_1}}{EI} = \frac{a_2 \overline{x_2}}{EI} + \frac{a_3 \overline{x_3}}{EI}$$

$$a_1 \overline{x_1} = a_2 \overline{x_2} + a_3 \overline{x_3}$$



BMD by Parts

18 R_B x 4 = (40 x 4.67) + (10 x 5.33) = 240.10

R_B = 3.33 KN

Reaction at A = Total load – R_B

R_A = (5 + 5 – 3.33) = 6.67 kN

Draw SFD

Bending Moment

M_B = 0
M_C = R_B x 2 = 3.33 x 2 = 6.66 kN.m
M_D = R_B x 4 – (5x2) = 3.33 x 4 – 5 x 2
Mx = 3.32 kN.m
M_A = R_B x ℓ – (5x4) – (5x2) = 3.33 x 6 – 5 x 4 – 5 x 2
M_A = –10.02 KN.m
Draw BMD

Problem 8
A Cantilever beam of 6m length Propped at 2m from free end it carries an udl of 12 kN/m over its entire span. Analyse the beam using area moment method Draw SFD and BMD indicating the values of salient points.

Solution

Span ℓ	= 4 m
Projection	= 2 m
udl w	= 12 KN/m
let R_A	= Reaction at A
R_B	= Prop. reaction

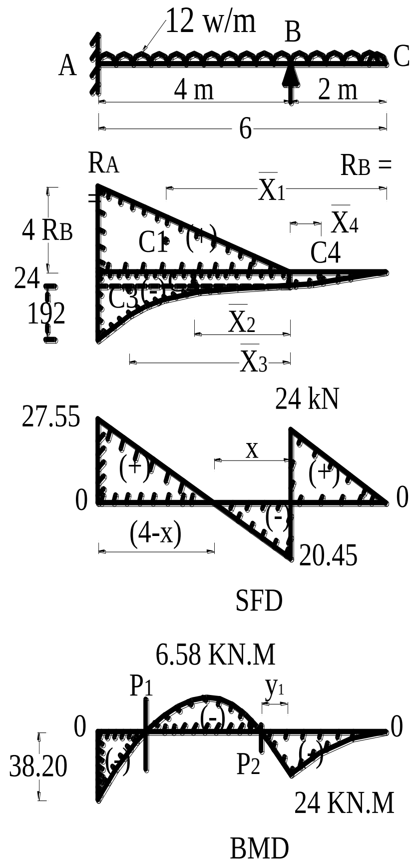
i) Bending moment by parts

R_B = Prop. Reaction
BM at A due to R_B x ℓ = 4 R_B
BM at A due to udl = $\frac{12 \text{ x } 6 \text{ x } 6}{2}$ = 216 KN.m
BM at B due to udl = $\frac{12 x 2^2}{2}$ = 24 KN.m
Draw BMD by parts

ii) Area

a_1 = area of BMD due to R_B

$a_1 \quad = \frac{1}{2}$ x 4 x 4 R_B = 8 R_B



Centroid

$\overline{x_1} = \frac{2}{3} x 4$ = 2.67 m from B

a_2 = area of BMD due to udl

$a_2 \quad = 4 \times 24 \quad = 96$ KN.m^2

Centroid of BMD from B

$\overline{x_2} \quad = \left(\frac{4}{2}\right) \quad = 2$m

$a_3 \quad = \frac{1}{3}$ x 4 x 192 $\quad$ = 256 kNm^2

$\overline{x_3} \quad = \frac{3}{4}$ x 4 $\quad$ = 3 m

$a_4 \quad = \frac{1}{3}$ x 2 x 24 $\quad$ = 16 kNm^2

$\overline{x_4} \quad = \frac{1}{4}$ x 2 $\quad$ = 0.5 m from B

ii) Prop. Reaction

By Mohr's Theorem – II

Upward deflection due to Prop. reaction = Downward deflection due to load

(Are of +ve Bm $x\ x$) = Area of –ve Bm $x\ x$)

$$\frac{a_1 \overline{x_1}}{EI} + \frac{a_4 \overline{x_4}}{EI} = \frac{a_2 \overline{x_2}}{EI} + \frac{a_3 \overline{x_3}}{EI}$$

$$a_1 \overline{x_1} + a_4 \overline{x_4} = a_2 \overline{x_2} + a_3 \overline{x_3}$$

(8 RB x 2.67) + (16 x 0.5) = (96 x 2) + (256 x 3)

21.36 R_B + 8 = 192 + 768 = 960

21.36 R_B = 952

$R_B = \frac{952}{21.36}$

R_B = 44.45 KN

Reaction at A = Total load – R_B

R_A = (12 x 6) – 44.45 = 27.55 kN

Maximum BM will occur at 'x' distance from C

$\frac{x}{20.45} = \frac{4-x}{27.55}$

27.55 x = 20.45 (4 – x)

27.55 x = 81.80 – 20.45x

48x $\quad$ = 81.80

$x = \dfrac{81.80}{48}$ = 1.70 m from B

Bending Moment

$M_C = 0$

$M_B = -12 \times \dfrac{2^2}{2}$ = 24 kN.m

$Mx = 44.45 \times 1.70 - 12 \times \dfrac{3.7^2}{2}$ = 6.58 kN.m

$M_A = R_B \times \ell - \dfrac{w.\ell^2}{2} = 44.45 \times 4 - 12 \times \dfrac{6^2}{2}$ = – 38.20 KN.m

Draw SFD & BMD

Problem 9

A beam of length 8m is fixed at one end and supported on a rigid prop. at the other end at the same level. The beam carries an udl of 8 kN/m over its entire length. Determine the prop. reaction and draw SFD and BMD.

Solution

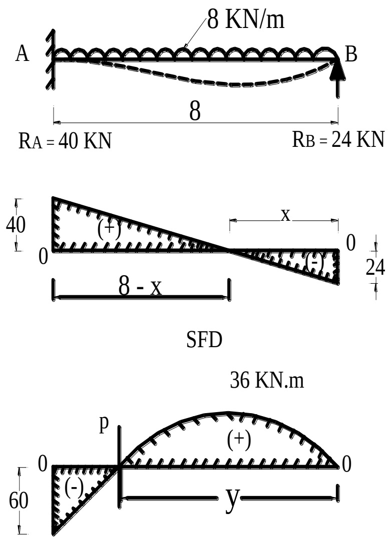
(i) Prop. reaction

$R_B = \dfrac{3}{8} w \times \ell = \dfrac{3}{8} \times 8 \times 8$

$\boxed{R_B = 24}$ kN

$R_A = (8 \times 8) - 24$

$\boxed{R_A = 40}$ kN



Draw SFD as shown in fig. 1.48

Max. (+ve) BM will occur at 'x' distance from Prop.

$\dfrac{x}{24} = \dfrac{8-x}{40}$

$40x = 24(8 - x)$

$40x = 192 - 24x$

$64x = 192$

$\boxed{x = 3}$ m

Bending Moment

$M_B = 0$

$M_A = (24 \times 8) - 8 \times 8 \times \dfrac{8}{2}$ = – 64 kN.m

$M_x = (24 \times 3) - 8 \times 3 \times \dfrac{3}{2}$ = 36 kN.m

Draw BMD

Point of Contra Flexure (P) :

Point of contra flexure will occur at 'y' distance from B.

$M_y = 0$

$R_B \times y - w\left(\frac{y^2}{2}\right) = 0$

$24 \times y - 8 \times \frac{y^2}{2} = 0$

$4.y^2 = 24\, y$

$y = \frac{24}{4}$

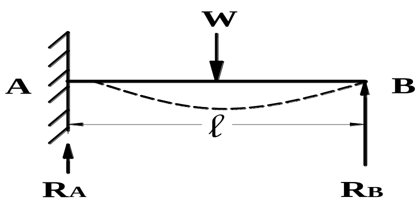
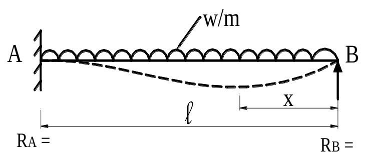
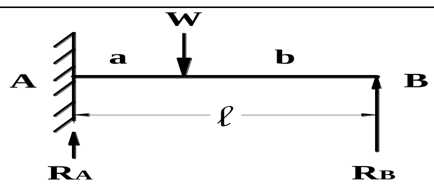
$\boxed{Y = 6 \text{ m}}$ from B

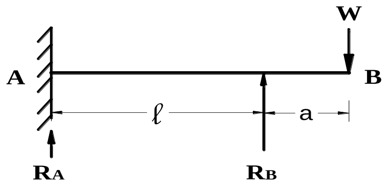
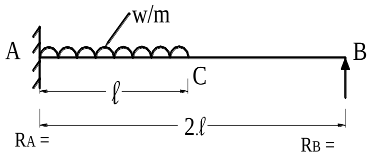
1.2 HIGHLIGHTS

1. Propped cantilever beam

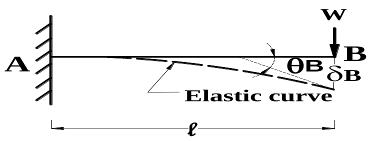
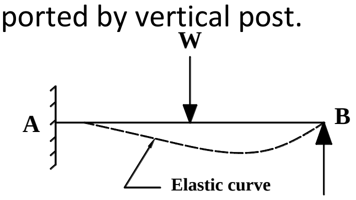
When a cantilever beam is supported by vertical post at free end (or) near the free end is called Propped cantilever beam. It is statically determinate beam and its degree of indeterminacy is 1.

2. Prop. Reaction

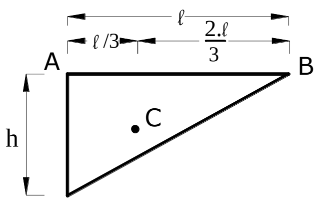
Sl. No.	Type of beam	Prop. Reaction (R_A & R_B)
1.		$R_B = \frac{5}{16} W$ $R_A = \frac{11}{16} W$
2.		$R_B = \frac{3}{8} \omega.\ell$ $R_A = \frac{5}{8} \omega.\ell$ Position max. Bm. $x = \frac{3}{8} \ell$
3.		$R_B = \frac{W.a^2}{2.\ell^3}(3\ell - a)$ $R_A = (W - R_B)$

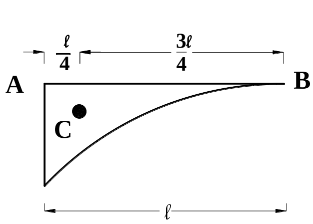
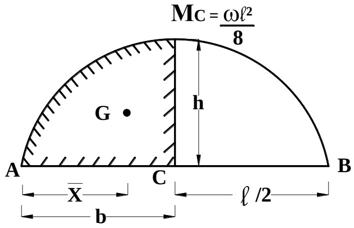
4.		$R_B = \dfrac{W}{2.\ell}(3\ell + a)$
5.		$R_B = \dfrac{7}{64}\,\omega.\ell$

3. Cantilever and Propped cantilever beam.

Sl. No	Cantilever beam	Propped cantilever beam
1.	One end fixed and another end free. 	One end fixed and another end supported by vertical post. 
2.	It is a statically determinate beam	It is statically indeterminate beam.
3.	Degree of indeterminacy is zero at fixed end is more.	Degree of indeterminacy is one End support moment is zero.
4.	There is no positive moment.	There is a positive moment also.
5.	Slope and deflection are more at free end.	Slope and deflection is zero at propped end.

4. Centroid and Area for standard cases.

Sl.No.	Shape	Centroid	Area
1.		$\bar{x}_A = \dfrac{\ell}{3}$ from 'A' $\bar{x}_B = \dfrac{2}{3} \times \ell$ from 'B'	A = ½ x ℓ x h

2.		$\bar{x}_A = \frac{\ell}{4}$ from 'A' $\bar{x}_B = \frac{3\ell}{4}$ from 'B'	$A = \frac{1}{3} x \, \ell \, x \, h$
3.		$\bar{x}_A = \frac{5}{8} x \, b$ from 'A' $\bar{x}_C = \frac{3}{8} x \, b$ from 'C'	$A = \frac{1}{3} x \, b \, x \, h$

1.2 QUESTIONS

One mark Questions

1. Define – A Prop
2. What is the degree of Indeterminacy of a propped cantilever beam?
3. Where the bending moment is maximum in a propped cantilever subjected to udl throughout.
4. Find the prop action of a propped cantilever beam subjected to point load at centre.
5. Draw the deflected shape of the any one-type of beam
6. What is the degree of indeterminacy of a Propped Cantilever beam?
7. State the prop reaction value of a propped cantilever beam with central point loaf 'W'
8. State the degree of indeterminacy of a fixed beam
9. What will be the degree of indeterminacy of a propped Cantilever? (unit 1.2)
10. What is the degree of indeterminacy of a fixed beam? (unit 1.2)

Three/Five mark Questions

1. Find the prop reaction of a propped cantilever beam subjected to a point load at mid-span by area Moment Method.
2. Draw the deflected shapes of cantilever beam, simply supported beam, propped cantilever beam, fixed beam and continuous beam
3. Find the prop reaction of a propped cantilever beam subjected to UDL throughout the span by area moment method.

Ten mark Questions

1. A beam of length 6m is fixed at one end and supported by a rigid prop at the other end at the same level. It carries an UDL of 10KN/m for a length of 4m from the fixed end. Determine the prop reaction and draw SFD and BMD.
2. A cantilever loaded with a point load at center of the span is propped at the free end. Find the fixed support moment and prop reaction.

3. A propped cantilever of length 8m is fixed at one end and supported on a rigid prop at the other end. It carries a point load of 40 kN at a distance of 5m from the fixed end. Determine the prop reaction. Draw SFD and BMD.
4. A beam of length 6m is fixed at one end and supported by a rigid prop at the other end. It carries an UDL of 5 kN/m for a length of 4m from the fixed end. Determine the prop reaction and draw SFD and BMD.
5. A beam of length 6m is fixed at one end and supported by a rigid prop at the other end. It carries an UDL of 30 kN/m over its length. Determine the prop reaction and draw SFD and BMD.
6. A Propped Cantilever of span 6m carries two equal point loads of 5kN act at 2m and 4m from left support. Determine the prop reaction and draw SFD and BMD
7. A beam of length 8m is fixed at one end and supported on a rigid unyielding prop carries an udl of 4 kN/m throughout the length. Draw SFD and BMD.
8. A cantilever of 6 m length is propped at 2m from the free end. It carries an udl of 12 kN/m throughout it length. Analyse the beam using area moment method and draw the SF and BM diagrams indicating the values at salient points.
9. Construct SFD and BMD for a propped cantilever of length 5m. with end prop carrying two point loads of 5KN, 10KN at 2m. and 3m. distances respectively from the fixed end.
10. A horizontal beam of 8 metre span is fixed at one end and simply supported at the other end. It carries an u.d.l. of 12 kN/m for 6m length from the fixed support. Determine the reactions at the supports and draw the SF and BM diagrams using area moment method. Determine also the positions of Zero moments in the beam.
11. A cantilever beam is supported by vertical post at free end 4m long and it carries a point load of 20 kN at centre and udl of 5 kN/m over entire length. Draw SFD and BMD.

2.1. FIXED BEAMS – AREA MOMENT METHOD

Introduction to fixed beams:

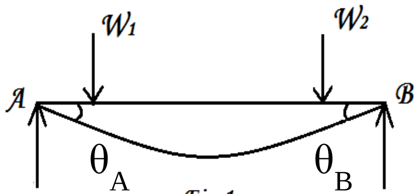


Fig 1

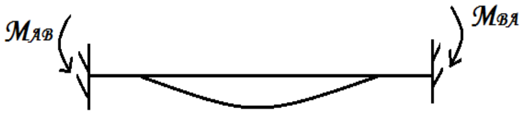


Fig 2

$$\theta_A = 0 \qquad \theta_B = 0$$

Fig 1 shows a simply supported beam AB carrying an external Load system.
Due to the load system a clockwise rotation at A (θ_A) and an anticlockwise rotation at B (θ_B) are developed.

To make these rotations (θ_A & θ_B) zero, an anticlockwise moment M_{AB} at A and a clockwise moment M_{BA} at B are to be applied.

These moments (M_{AB} & M_{BA}) can be developed by fixing the supports A & B.

These end moments are called "Fixed end moments" and such a beam is called a "Fixed beam".

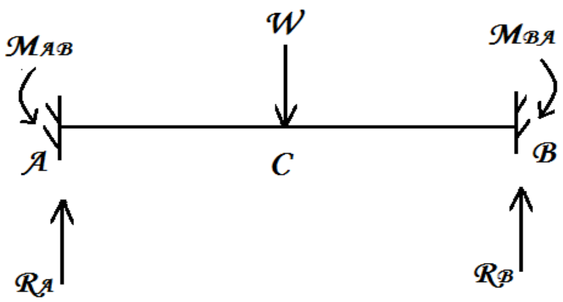
Advantages:

1) The fixed end moments reduce the max bending moment near the mid span.
2) Smaller c/s and hence economical.
3) Less deflection
4) Stiffer, stronger and stable.

Disadvantages:

1) Being an indeterminate structure, additional equations, besides static equilibrium equation are necessary for the analysis.
2) Proper care should be taken for the effects due to temperature and secondary stresses.

Degree of indeterminacy of Fixed beam:



No of unknown reactions = 4

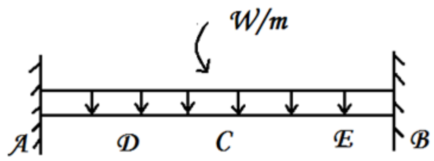
(R_A, R_B, M_{AB} and M_{BA})

No of available static equilibrium equation = 2

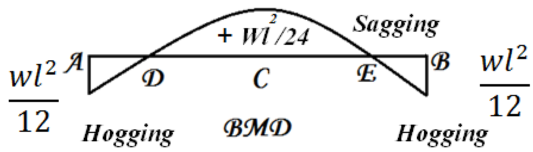
Redundant reaction (excess unknown) = **2**(4-2)

∴ Degree of indeterminacy is **2.**

Sagging and Hogging moments:



Fixed beam with u.d.l



Here point D & E are Point of contra - flexure

In case of fixed beams, both hogging bending moment and sagging bending moments are present.

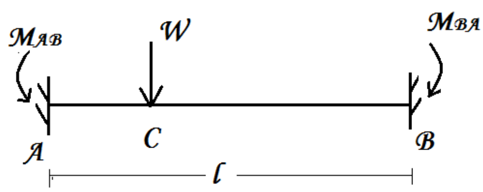
The fixed end moments are usually hogging in nature.

The mid span moment (for symmetrical loading) is sagging in nature.

At the ends, the moments are hogging and gradually reduces to zero at the point of contra-flexure and gradually increases to max sagging moment and then sagging moment gradually reduces to reach another point of contra-flexure and after that increasing to hogging moments at other end.

Thus there are two point of contra-flexure.

Determination of Fixed end moments by Area moment method:



The fixed ends can be calculated by Area moment method which uses Mohr's theorems.

In case of symmetrical loading, the fixed ends are equal (i.e.) MAB = MBA. For this case, the fixed end moments can be calculated by applying Mohr's theorem I.

In case of unsymmetrical loading, the fixed end moments are not equal. (i.e.) MAB is not equal to MBA. For this case, the fixed end moments can be calculated by both the Mohr's theorem.

(i.e.) Mohr's theorems I and II.

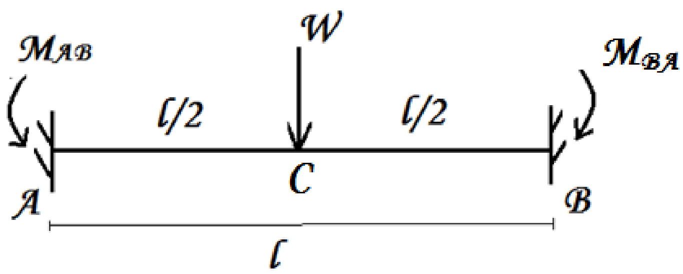
Derivations of expression for standard cases of fixed beams:

The following are the standard cases

I. Fixed beam with central point load (symmetrical loading).
II. Fixed beam with UDL throughout (symmetrical loading).
III. Fixed beam with a non-central (eccentric) point load (unsymmetrical loading).

Standard case I

FIXED BEAM WITH CENTRAL POINT LOAD:



Notations:

W – Central point load.

L – Span of the beam.

M_{AB} – Fixed end moments at A.

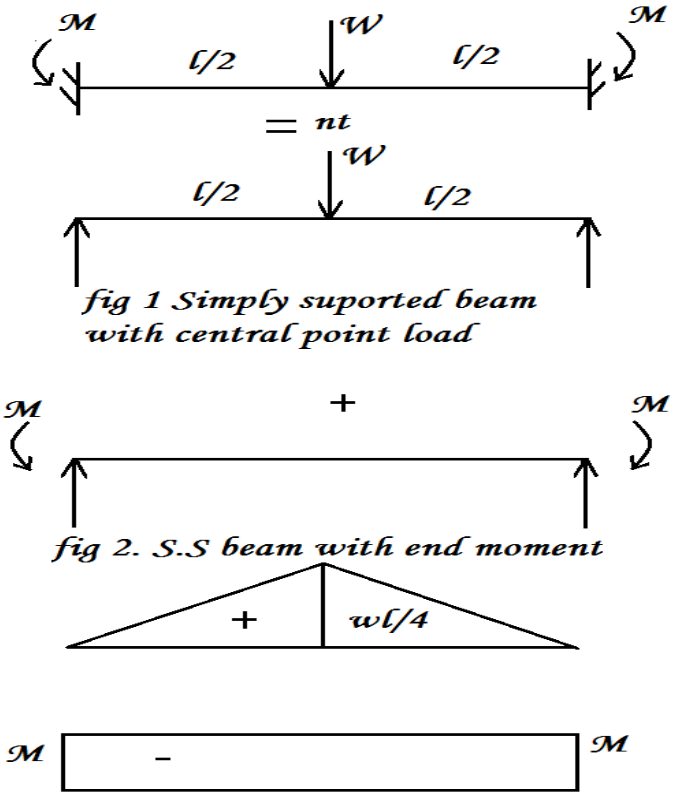
M_{BA} – Fixed end moments at B.

Concept:

* This standard case is a case of **symmetrical loading.**
* The fixed end moments are equal.
  (i.e.) $M_{AB} = M_{BA} = M$ (say).
* The fixed end moments (M_{AB} & M_{BA}) are evaluated by applying **Mohr's theorem I**.

Approach:

The given Fixe



μ diagram (free BMD)

μ' diagram (fixed BMD)

In the figure,

The BMD's for the equivalent S.S beams are drawn.

The BMD for the S.S beam with loading is called as μ diagram or free BMD.

For this case the μ diagram is a triangle with a central ordinate of Wl/4.

The BMD for S.S beam with ends moments is called μ'diagram or fixed BMD.

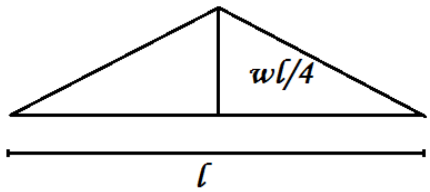
For this case, the end moments are equal and hence μ' diagram is a rectangle.

The end moments are calculated by equating,

Area of μ diagram = Area of μ' diagram.

<u>Derivation:</u>

Step 1: Area of μ diagram



Area of μ diagram

= Area of triangle

= ½ bh

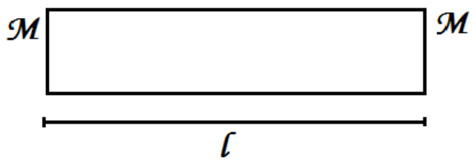
= ½×l×Wl/4

= $Wl^2/8$

Step 2: Area of μ' – diagram:

Area of μ' diagram

= Area of rectangle



= l × b

= l × M

= Ml

Step 3: Fixed end moments:

Area of μ' diagram = Area of μ diagram

Ml = $Wl^2/8$

M = Wl/8

(i.e.) $\mathbf{M_{AB} = M_{BA} = M = Wl/8.}$

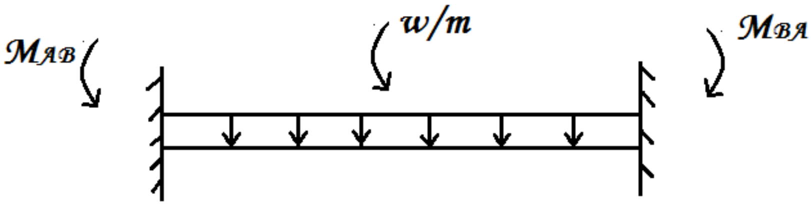
<u>Result:</u>

The fixed end moments

$\mathbf{M_{AB} = M_{BA} = Wl/8.}$

Standard case II

FIXED BEAM WITH UDL THROUGOUT THE SPAN



Notations:

w - UDL throughout the span

l -Span of the beam

M_{AB} - fixed end moment at A

M_{BA} - fixed end moment at B

Concept:

- This standard case is a case of **symmetrical loading.**
- The Fixed end moments are equal.
  (i.e.) $M_{AB} = M_{BA} = M$ (say)
- The fixed end moments (M_{AB} & M_{BA}) are evaluated by applying **Mohr's Theorem I.**

Approach:

The given fixed beam is converted to an equivalent S.S beam.

From the diagram,

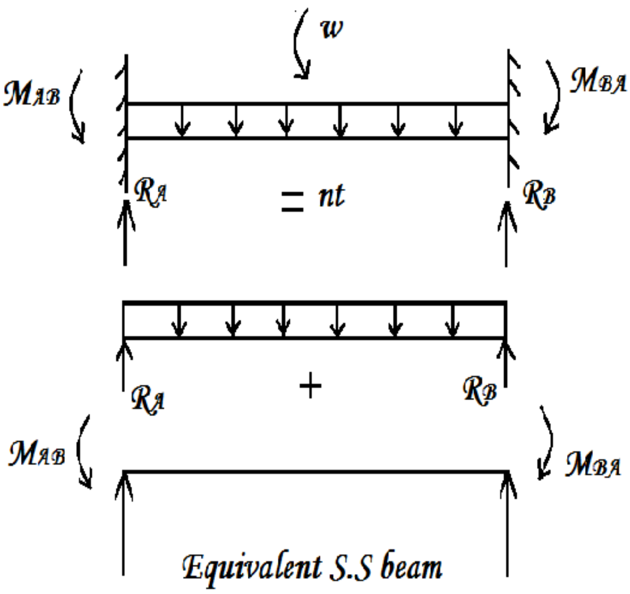
The BMD's for equivalent S.S beams are drawn.

The μ diagram (free BMD) is a second degree parabola with a central ordinate of wl2/8.

The μ' diagram (fixed BMD) is a rectangle with length 'l' and breadth 'M'.

The end moments are calculated by equating,

Area of μ' diagram = Area of μ diagram.



Step 1: Area of μ diagram:

Area of μ diagram

= Area of parabola

= 2/3 bh

= $2/3 \times l \times wl^2/8$

= $wl^3/12$.

Step 2: Area of μ' diagram:

Area of μ' diagram

= Area of rectangle

= $l \times b$

= $l \times M$

= Ml

Step 3: Fixed end moments:

Equating,

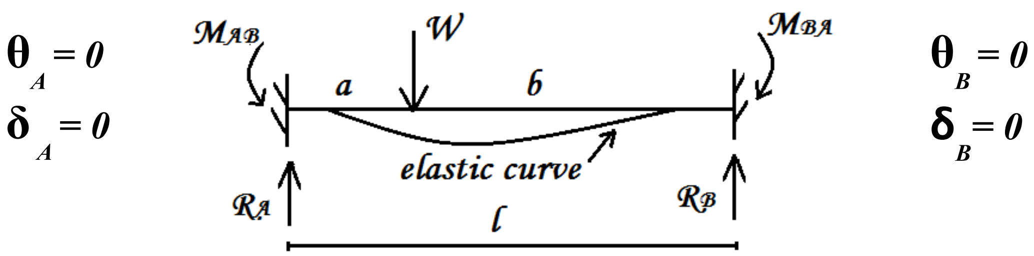
Area of μ' diagram = Area of μ diagram

$l \times M = wl^3/12$

Therefore $M = wl^2/12$

$$\mathbf{M_{AB} = M_{BA} = M = wl^2/12}$$

FIXED BEAMS WITH UNSYMMETRICAL LOADING



Example fig

In the fig,

$a \neq b$; $M_{AB} \neq M_{BA}$;

$R_A \neq R_B$.

Boundary condition,

$\theta_A = 0$; $\theta_B = 0$; $\delta_A = 0$; $\delta_B = 0$;

When a fixed beam is subjected to unsymmetrical loading,

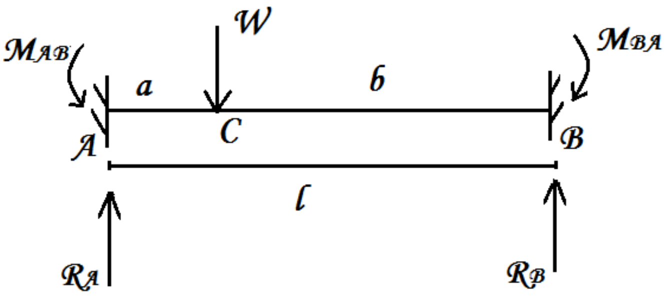
- Fixed end moments (MAB & MBA) are not equal (i.e.) MAB ≠ MBA.
- The vertical reactions (RA & RB) are also not equal.
- (i.e.) RA ≠ RB.
- Since the degree of indeterminacy or redundancy is two, the redundant reactions (excess unknown reactions) are two.
- Hence we require two more equations in addition to the two static equilibrium equations.
- The two additional equations can be obtained from the geometric conditions of the elastic curve at the ends (boundary conditions).
- Here we use the conditions that the slope and deflection at the ends are zero.
- (i.e.) θA = 0, δA = 0; (or) θB = 0, δB = 0;

- We make use of the area moment theorem (Mohr's theorem) I & II between the two ends to obtain the two equations.
- The first equation is obtained from theorem I making use of the condition that the slope at the ends of the beam are zero.
- The second equation is obtained from theorem II making use of the condition that the deflection at the ends are zero.

  Hence from theorem I
  $\sum a_{AB}/ EI= 0$
  $EI \neq 0$, therefore
  $\sum a_{AB} = 0$ - (1)
  Hence from theorem II
  $$\frac{\sum a_{AB}\overline{X}_A}{EI}$$
  $EI \neq 0$, $\sum a_{AB} \times \overline{X}_A = 0$

Standard case III

FIXED BEAM WITH A NON - CENTRAL (ECCENTRIC) POINT LOAD



Notations:

W - Non – central point load
M_{AB} - Fixed end moment at A
M_{BA} - Fixed end moment at B
R_A - Vertical reaction at A
R_B - Vertical reaction at B
l - Span of the beam

Concept:

- This standard case is a case of unsymmetrical loading.
- The Fixed end moments are not equal. (i.e.) $M_{AB} \neq M_{BA.}$
- The Fixed end moments are evaluated by applying Mohr's theorems I & II.

Approach:

The given fixed beam is converted into equivalent simply supported beams and the corresponding μ diagram (Free BMD) and μ' diagram (Fixed BMD) are drawn.

The μ diagram is a triangle with an ordinate of Wab/l at C.

Since MAB≠ MBA, the μ' diagram is trapezium. The fixed end moments are evaluated by applying Mohr's theorems I & II.

From theorem I,

Area of μ' diagram = Area of μ diagram

From theorem II,

Moment of area of μ diagram about A = Moment of area of μ'

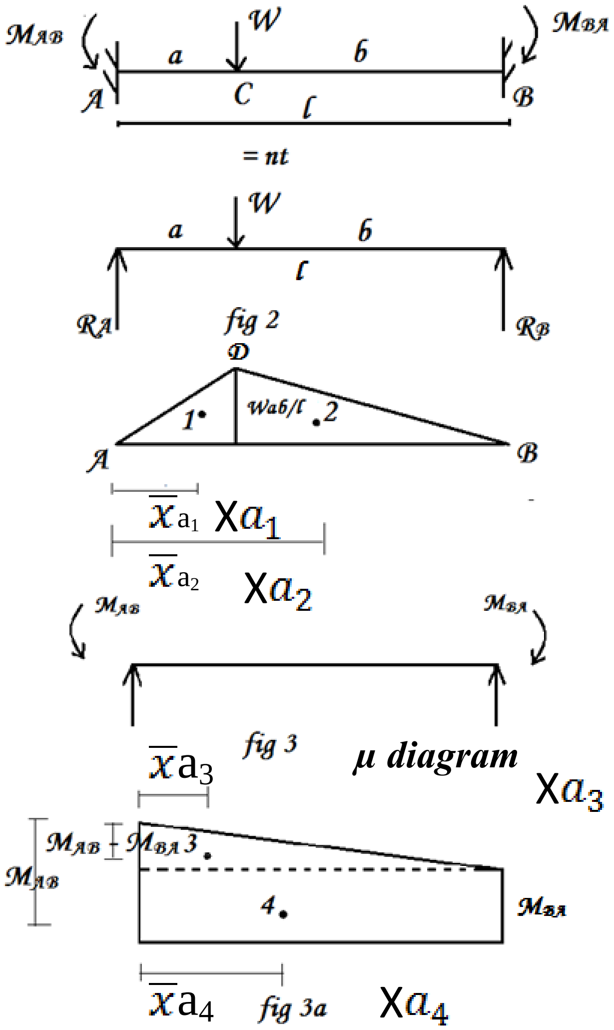
Diagram about B

In this fig

Fig 2 & fig 3 are equivalent S.S beams

Fig 2(a) µ diagram (Free BMD)

Fig 3(a) µ' diagram (Fixed BMD)



Derivation:

Step I: Area of µ diagram:

Area of µ diagram = Area of triangle

$= \frac{1}{2} bh$

$= \frac{1}{2} \times l \times \frac{Wab}{l}$

$= Wab/2$

Step II: Area of µ' diagram:

Area of µ' diagram = Area of trapezium

= Area of rectangle + Area of triangle

$= lb + \frac{1}{2} bh$

$= (l \times M_{BA}) + \frac{1}{2} \times l \times (M_{AB} - M_{BA})$

$= l \times M_{BA} + \frac{1}{2} \times l \times M_{AB} - \frac{1}{2} \times l \times M_{BA}$

$= \frac{1}{2} \times l \times M_{BA} + \frac{1}{2} \times l \times M_{AB}$

$= (M_{AB} + M_{BA}) \times l/2$

Step III: Application of Mohr's theorem I:

$\frac{\sum a_{AB}}{EI} = 0$, $EI \neq 0$

$\therefore \sum a_{AB} = 0$

Area of μ diagram – Area of μ' diagram = 0

Area of μ diagram = Area of μ' diagram

$Wab/2 = (M_{AB} + M_{BA})\, l/2$

$\therefore M_{AB} + M_{BA} = Wab / l$ - (1)

Step IV: Moment of area of μ diagram:

Moment of area of μ diagram about the left end A

$= a_1 . \bar{x}a_1 + a_2 . \bar{x}a_2$

$a_1 = \frac{1}{2} \times a \times \frac{Wab}{l}$

$\bar{x}a_1 = \frac{2}{3}\, a$

$a_1 . \bar{x}a_1 = \frac{1}{2} \times a \times \frac{Wab}{l} \times \frac{2}{3} \times a$

$= Wa^3b / 3l$

$a_2 = \frac{1}{2} \times b \times \frac{Wab}{l}$

$\bar{x}a_2 = (a + b/3) = \left(\frac{3a+b}{3}\right)$

$a_2 . \bar{x}a_2 = \frac{1}{2} \times b \times \frac{Wab}{l} \times \left(\frac{3a+b}{3}\right)$

$= \frac{Wab^2}{6l}\, (3a + b)$

$\therefore$ Moment of area of μ diagram about A

$= a_1 . \bar{x}a_1 + a_2 . \bar{x}a_2$

$= \frac{Wa^3b}{3l} + \frac{Wab^2}{6l} \times (3a + b)$

$= \frac{Wab}{6l}\, (2a^2 + 3ab + b^2)$

Step V: Moment of area of μ' diagram about A

Moment of area of μ' diagram about A,

$= a_3 . \bar{x}a_3 + a_4 . \bar{x}a_4$

$a_3 = ½\, bh$

$= ½ \times l \times (M_{AB} - M_{BA})$

$= (M_{AB} - M_{BA}) \times l/2$

$\bar{x}a_3 = l/3$

$a_3 . \bar{x}a_3 = (M_{AB} - M_{BA}) \times (l/2) \times (l/3)$

$= (M_{AB} - M_{BA}) \times l^2/6$

$a_4 = M_{BA} \times l$

$\bar{x}a_4 = l/2$

$a_4 . \bar{x}a_4 = M_{BA} \times l \times l/2$

$= \frac{M_{BA}}{2}\, l^2$

$\therefore a_3 . \bar{x}a_3 + a_4 . \bar{x}a_4 = \frac{M_{AB} - M_{BA}}{6}\, l^2 + M_{BA}\frac{l^2}{2}$

$= \frac{M_{AB} - M_{BA} + 3M_{BA}}{6} \times l^2$

$= \frac{M_{AB} + 2M_{BA}}{6}\, l^2$

$\therefore$ Moment of area of μ' diagram about A

$\frac{M_{AB} + 2M_{BA}}{6}\, l^2$ - (2)

Step VI: Fixed end moments:

$(M_{AB} + 2\ M_{BA}) \times l^2/6 = \frac{Wab}{6l}\ (2a^2 + 3ab + b^2)$

$M_{AB} + 2\ M_{BA} = \frac{Wab}{l^3}\ (a^2 + ab + a^2 + 2ab + b^2)$

$= \frac{Wab}{l^3}\ [a\ (a+b) + (a+b)^2]$

$= \frac{Wab}{l^3}\ (al + l^2)$

$= \frac{Wab}{l^3}\ l\ (a+l)$

$M_{AB} + 2\ M_{BA} = \frac{Wab}{l^2}\ (l+a)$ - (3)

$M_{AB} + M_{BA} = \frac{Wab}{l}$

Equating 1 & 3

$M_{BA} = \frac{Wab}{l^2}\ (l+a) - \frac{Wab}{l}$

$= \frac{Wab}{l} + \frac{Wa^2b}{l^2} - \frac{Wab}{l}$

$\therefore M_B = \frac{Wa^2b}{l^2}$ (Hogging)

Substituting in 1

$M_{AB} + \frac{Wa^2b}{l^2} = \frac{Wab}{l}$

$M_{AB} = \frac{Wab}{l} - \frac{Wa^2b}{l^2}$

$= \frac{Wab\ l - Wa^2b}{l^2}$

$= \frac{Wab\ (l-a)}{l^2}$

$= \frac{Wab^2}{l^2}$ (since l-a = b)

$\therefore M_{AB} = \frac{Wab^2}{l^2}$ (Hogging)

Results:

Fixed end moments

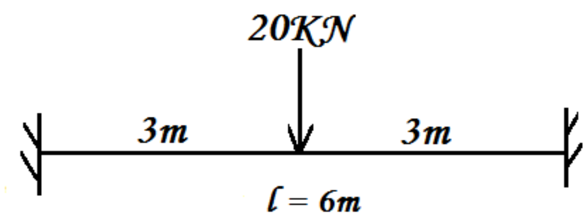
$M_{AB} = \frac{Wab^2}{l^2}$ (hogging).

$M_{BA} = \frac{Wa^2b}{l^2}$ (hogging).

Illustrative examples for symmetrical loading:

(1). A Fixed beam of span 6m carries a central point load of 20kN. Analyse the beam for shear, BM and draw the SFD & BMD.

Given data:



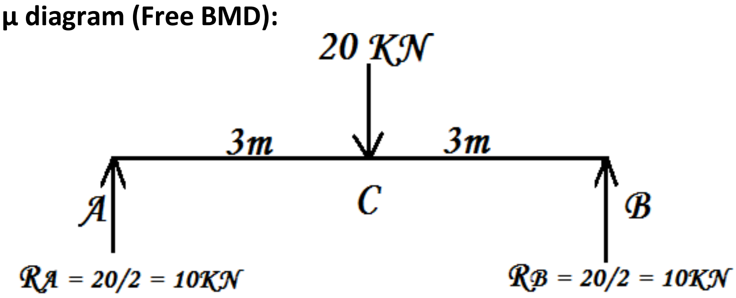
To find:

To analyse the beam for shear & BM

Solution:

Consider M_{AB} & M_{BA} as redundant reactions. (Excess unknowns)

Step 1: μ diagram (Free BMD):



$R_A = R_B = \frac{total\ load}{2}$

$= 20 / 2 = 10kN$

$M\mu_A$ = 0 (simple support)

$M\mu_C = + R_A \times 3 - 20 \times 0 = + 10 \times 3$

$= + 30kNm$

$M\mu_B$ = 0 (simple support)

Complete μ diagram.

Step 2: Fixed end moments: (M_{AB} & M_{BA})

By symmetry $M_{AB} = M_{BA} = M$

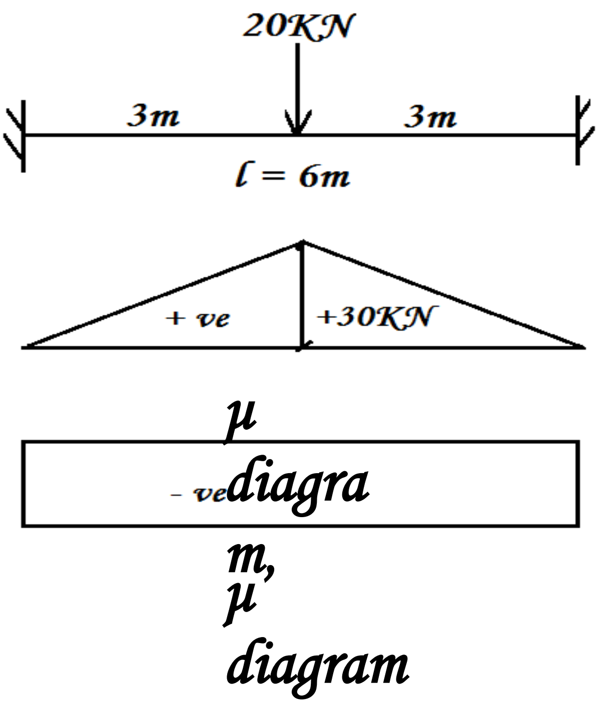
For symmetrically loaded fixed beam

Area of μ' diagram = Area of μ diagram

$M \times 6 = ½ \times 6 \times 30$

$= 90kNm.$

$M = 15kNm$



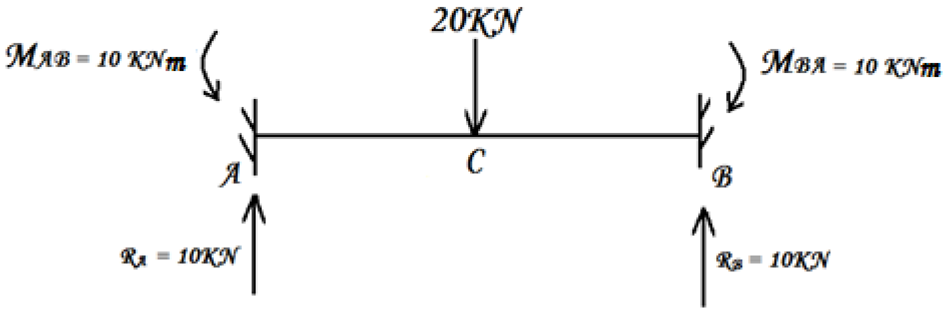
Step 3: Vertical reactions (R_A, R_B):

By symmetry,

$R_A = R_B = \frac{total\ load}{2}$

$= 20 / 2 = 10kN\ (\uparrow)$

Step 4: Shear force (V_x):



V_A (L) = 0

V_A (R) = + 10kN

V_C (L) = V_A (R) = +10kN

V_C (R) = V_C (L) – 20 = +10 – 20

= -10kN

V_B (L) = V_C (R) = - 10kN

V_B (R) = 0

Where L = left side, R = right side

Complete SFD.

Step 5: Bending moment (M_X):

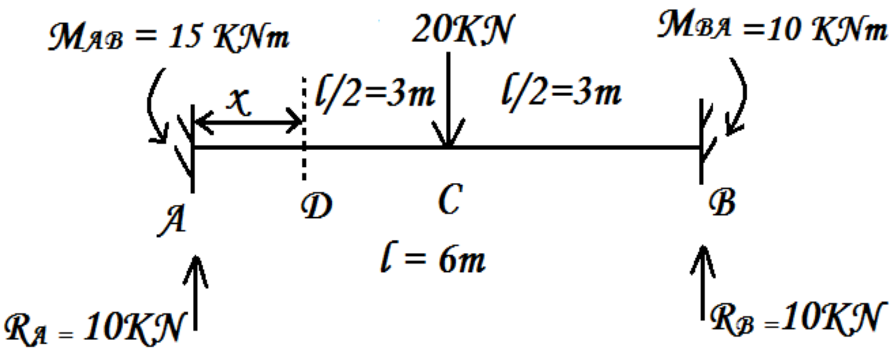
M_A = - M_{AB} (=M) = - 15kNm

M_C = - M + R_A × 3 = - 15 + (10 × 3)

= + 15kNm

M_B = - M_{BA}(= M) = - 15kNm

Points of contra – flexure:



Since the BM changes its sign from –ve to +ve from A to C and from +ve to –ve from C to B.

∴ There are two points of contra – flexure.

Let one of the point of contra – flexure be D at a distance X from A.

∴ M_D = 0

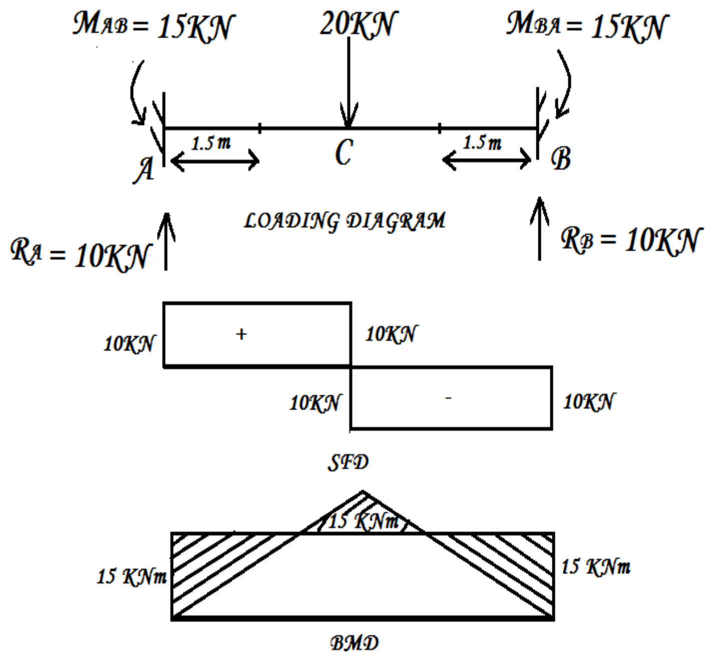
+ R_A × x – M_{AB} = 0

+ 10 × x – 15 = 0

∴X = 1.5m from A

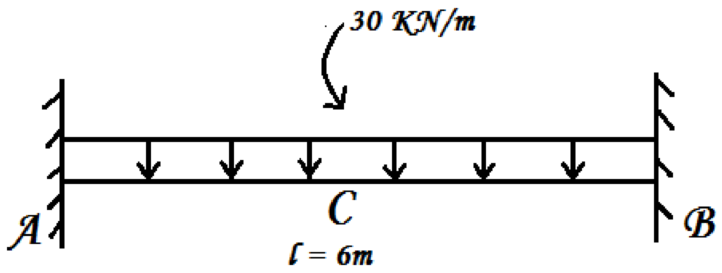
By symmetry another point of contra – flexure at 1.5m from B.

SFD & BMD



(2). A fixed beam of span 6m carries a UDL of 30kN/m throughout the span. Analyse the beam for shear, BM and draw SFD & BMD.

<u>Given data:</u>



Loading Diagram

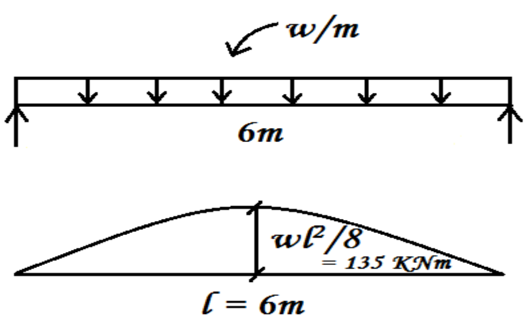
<u>To find:</u>

To analyse beam for shear, BM and draw SFD & BMD.

<u>Solution:</u>

The given fixed beam is a symmetrically loaded fixed beam.

Step 1: μ diagram (free BMD):



μ diagram

μ diagram is a second degree parabola with max value as $Wl^2/8$ at mid span and with ends 0.

Area of μ diagram = $\frac{2}{3} \times 6 \times 135$

= $540 kNm^2$

Step 2: μ' diagram (fixed BMD):

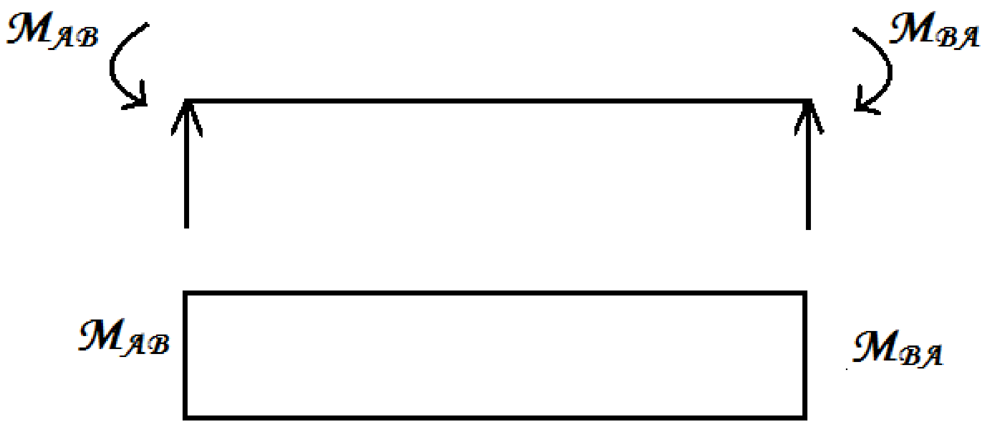
Since the loading is symmetrical.

$M_{AB} = M_{BA}$

The μ' diagram is a rectangle.

Area of μ' diagram = $M_{AB} \times l = M_{AB} \times 6$

= $6 M_{AB}$.



u'

Step 3: Redundant reactions (M_{AB} & M_{BA}):

Consider the fixed end moments MAB & MBA as redundant reactions (excess unknown)

For symmetrically loaded fixed beam

Area of μ' diagram = Area of μ diagram

$6 M_{AB}$. = 540

M_{AB}. = 540 / 6 = 90kNm.

By symmetry $M_{AB} = M_{BA}$

∴ $M_{AB} = M_{BA}$ = 90kNm (hogging).

Step 4: Vertical reactions (R_A & R_B):

By symmetry

$R_A = R_B = \frac{total\ load}{2}$

$= \frac{30 \times 6}{2}$

= 90kN.

Step 5: Shear force (V_X):

$V_A = + R_A = + 90kN$

$V_C = + R_A - wl = + 90 - (30 \times 3) = 0$

$V_B = - R_B = - 90kN$

Complete SFD.

Step 6: Bending moment (M_X):

$M_A = - M_{AB} = - 150kNm$

$M_C = + R_A \times (l/2) - M_{AB} - w \times (l/2) \times (l/4)$

$= + 90 \times (6/2) - 90 - [30 \times (6/2) \times (6/4)]$

= + 270 – 90 – 135 = 45kNm

M_B = - M_{BA} = - 150kNm

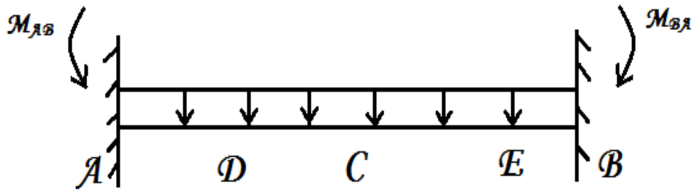
Point of contra – flexure:

For this standard case, the point of contra – flexure D & E are 0.211 l from the nearby support.

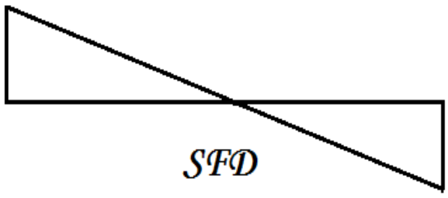
0.211 × 6 = 1.266m from either support.

Complete BMD.

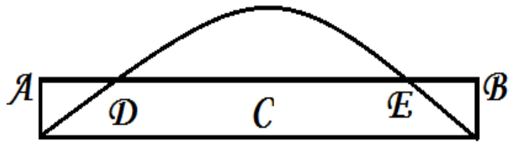
SFD & BMD:



Fixed beam with u.d.l



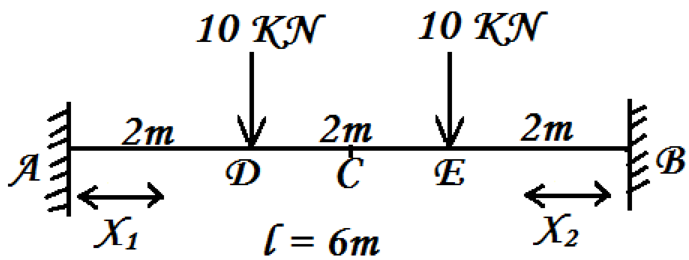
SFD



BMD

(3). A fixed beam of span 6m carries a point loads of 10kN at one – third points. Analyse the beam for shear, BM and draw the SFD & BMD.

Given data:

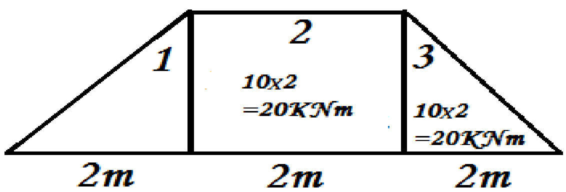


To find:

To analyse the beam for shear, BM and draw the SFD & BMD.

Solution:

Step 1: μ diagram



$a_1 = ½\ bh$

$= ½ \times 2 \times 20$

$= 20 kNm^2$

$a_2 = l \times b$

$= 20 \times 2 = 40\ kNm^2$

$a_3 = a_1 = 20\ kNm^2$

Area of μ diagram = $a_1 + a_2 + a_3 = 80 kNm^2$

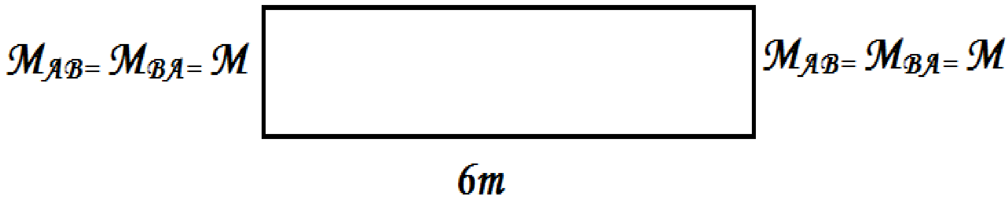
For concentrated loads (point loads) at 1/3rd points, the μ diagram is a trapezium.

Step 2: μ' diagram

By symmetry

$M_{AB} = M_{BA} = M$

∴ The μ' diagram is a rectangle.



Step 3:Redundant reactions (M_{AB}& M_{BA})

Consider M_{AB}& M_{BA} as redundants (excess unknowns)

But $M_{AB} = M_{BA} = M$ (By symmetry)

There is only one redundant reaction.

∴ By applying Mohr's theorem I,

Area of μ' diagram = Area of μ diagram

6 M = 80

M = 80/6 = 13.33kNm

∴$M_{AB} = M_{BA} = 13.33 kNm$

Step 4: Vertical reactions (R_A , R_B)

$R_A = R_B = \frac{total\ load}{2}$

$= \frac{10+10}{2} = 10 kN$

Step 5: Shear force (V_X)

$V_A (L) = 0$

$V_A (R) = + R_A = + 10kN$

$V_D (L) = V_A (R) = + 10kN$

$V_D (R) = V_D (L) – 10$

$= 10 - 10 = 0$

$V_E (L) = V_D (R) = 0$

$V_E (R) = V_E (L) – 10$

$= 0 - 10 = -10kN$

$V_B (L) = V_E (R) = - 10kN$

$V_B(R) = 0$

Where L = Left side, R = Right side

Complete SFD.

Step 6: Bending moment (M_X)

$M_A = - M_{AB} = - 13.33kNm$

$M_B = + R_A \times 2 – M_{AB}$

$= + 10 \times 2 – 13.33 = + 6.67kNm$

$M_C = + R_A \times 3 - M_{AB} - 10 \times 1$

$= + 10 \times 3 - 13.33 - 10 = -6.67\text{kNm}$

$M_E = M_D = + 6.67\text{kNm}$

$M_B = - M_{BA} = - 13.33\text{kNm}$

Points of contra – flexure:

Since the BM changes its sign –ve to +ve from A to D and +ve to –ve from E to B. There are two points of contra – flexure one in section AD and another in section EB.

By symmetry both will be equal distance from the nearby support.

Let the BM be 0 at a distance X from A.

$M_X = M_D = 0$

$+R_A \times x - M_{AB} = 0$

$+10 \times x - 13.33 = 0$

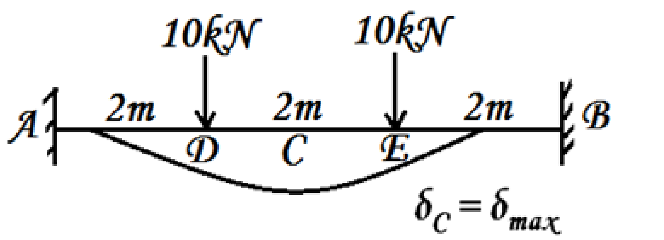
$10x = 13.33$

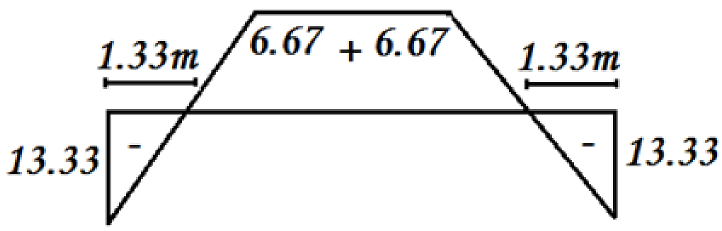
$x = 1.33\text{m}$

By symmetry,

$x_1 = 1.33\text{m}$ from A

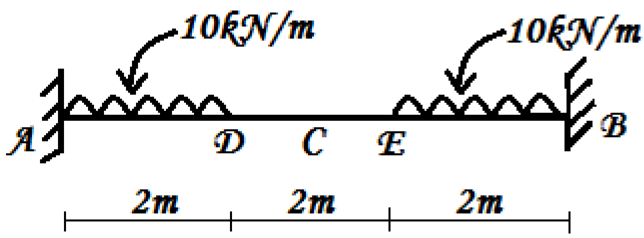
$x_2 = 1.33\text{m}$ from B





(4). A fixed beam of span 6m carries an u.d.l of 10kN/m run over a length of 1/3rd span from both the supports. Calculate the fixed end moments and draw SFD & BMD.

Given data:



Solution:

Step 1:Redundant reaction (M_{AB}, M_{BA})

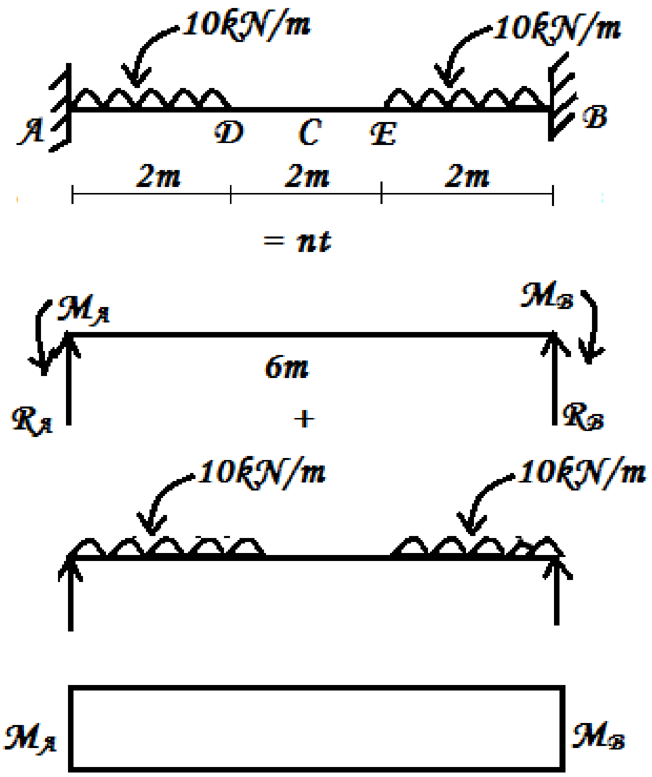
The given loading is symmetrical

$M_{AB} = M_{BA}$

Using $\Theta_B = 0$,

Area of μ diagram = Area of μ' diagram

μ' diagram:



μ' diagram

μ diagram

μ_A = 0kNm

μ_D = 20 × 2 – (10×2×2/2) = 20kNm

μ_C = 20 × 3 - 10×2× (2/2 + 1) = 20kNm

μ_E = 20 × 4 – 10×2×(2/2 + 2) = 20kNm

μ_B = 20 × 6 – 10×2×(2/2 + 2) – 10×2×2/2 = 0kNm

Complete μ diagram.3

Area of μ' diagram = Area of μ diagram

l × b = 2 (2/3 × bh) + (l×b)

M_{AB} × 6 = 2 (2/3 ×2×20) + (2×20)

6M_{AB} = 93.33

M_{AB} = 15.56kNm

M_{AB} = M_{BA}

M_{BA} = 15.56kNm

Step 2: Vertical reactions (R_A & R_B)

$R_A = R_B$. So,

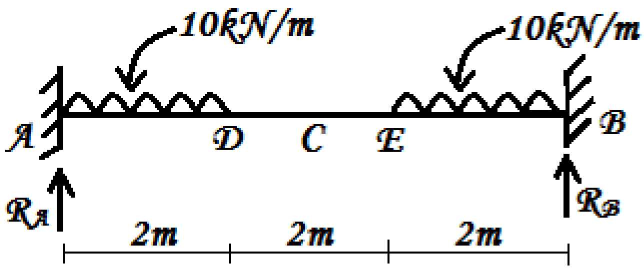
$R_A = \frac{total\ loads}{2}$

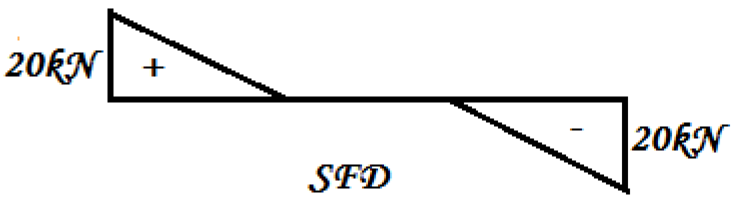
$= \frac{(20\times2)+(10\times2)}{2}$ = 40/2

R_A = 20kN

$R_A = R_B = 20kN$

Step 3: Shear force (V_X)





$V_A(L) = 0kN$

$V_A(R) =+ R_A =+ 20kN$

$V_D = +R_A – (10×2) = +20 – 20 = 0kN$

$V_C = 0kN$

$V_E = 0kN$

$V_B(L) = R_A – (10×2) – (10×2) = 20 – 20 – 20 = -20KN$

$V_B(R) = 0kN$

Point of zero shear:

Shear Force is 0 at D, C, and E.

$V_D = \quad V_C = V_E = 0kN.$

Complete SFD.

Step 4: Bending moment (M_X)

$M_A = - M_{AB} = - 15.56kNm$

$M_D = R_A × 2 - M_{AB} - 10×2×2/2 = 20 × 2 – 15.56 – 20$

$= 4.44kNm$

$M_C = R_A × 3 – M_{AB} – 10×2×(2/2 + 1) = 20×3 - 15.56 – 40$

$= 4.44kNm$

$M_E = R_A × 4 – M_{AB} – 10×2×(2/2 + 2) = 20×4 – 15.56 – 60$

$= 4.44kNm$

$M_B = R_A × 6 – M_{AB} - 10×2×(2/2 + 4) – (10×2×2/2)$

$= 20×6 – 15.56 – 60 – 20 = - 15.56kNm$

Point of contra - flexure:

$M_F = R_A × x – 15.56 – 10 × x × x/2 = 0$

$20x - 15.56 – 5x^2 = 0$

$5x^2 – 20x + 15.56 = 0$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+20 \pm \sqrt{(-20)^2 - 4×5×15.56}}{2×5}$$

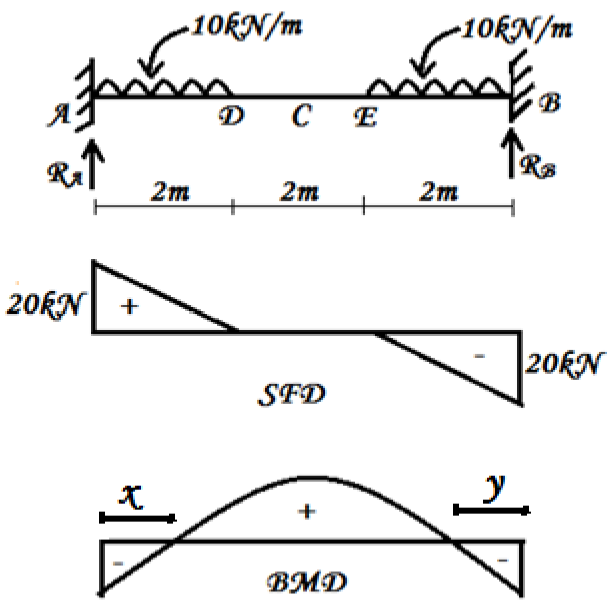
$$= \frac{+20 \pm 9.42}{10}$$

a = + 1.06m, b = + 2.94m

Take a distance of 1.06m as x.

x = y so, y = 1.06m

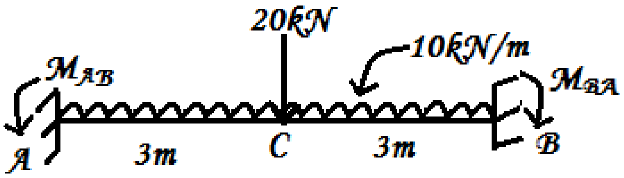
Complete BMD.



(Alternative Method using Formula)

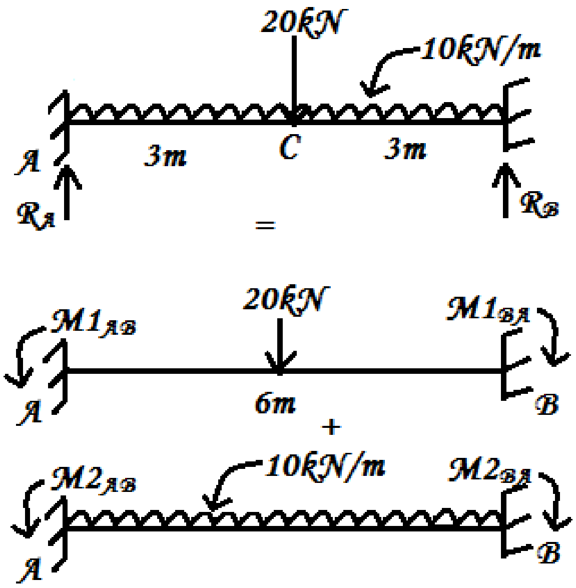
(5). A fixed beam of span 6m carries an u.d.l of 10kN/m throughout the span. It also carries a central point load of 20kN. Calculate the fixed end moments & draw SFD & BMD. Locate the point of contra flexure.

Given data:



Solution:

Step 1: Fixed end moments



$M_{AB} = M1_{AB} + M2_{AB}$

$= \frac{Wl}{8} + \frac{wl^2}{12}$

$= \frac{20\times6}{8} + \frac{20\times6^2}{12}$

$= 15 + 30 = 45kNm$

Loading is symmetrical.

$M_{AB} = M_{BA} = 45kNm$ (Hogging).

Step 2: Vertical reaction (R_A & R_B)

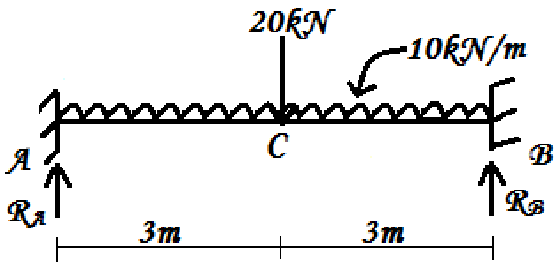
Loading is symmetrical

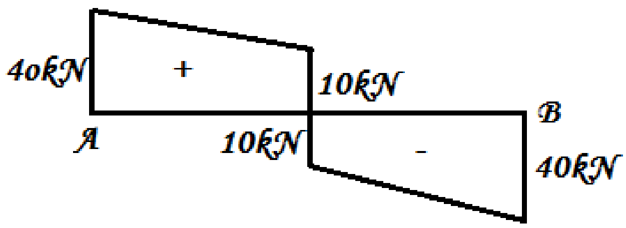
$R_A = \frac{Total\ loads}{2}$

$= \frac{20+(10\times6)}{2} = 40kN$

$R_A = R_B = 40kN$

Step 3: Shear force





$V_A(L) = 0kN$

$V_A(R) = 40kN$

$V_C(L) = 40 – (10\times3) = 10kN$

$V_C(R) = 40 – (10\times3) – 20 = - 10kN$

$V_B(L) = 40 – (10\times6) – 20 = - 40kN$

$V_B(R) = 0kN$

Point of zero shear:

S.F values suddenly changes exactly at C.

Point of zero shear is at C.

Complete SFD.

Step 4: Bending moment

$M_A = - 45kNm$

$M_C = - 45 + (40\times3) – (10\times3\times3/2) = +30kNm$

$M_B = - 45 + (40\times6) – (10\times6\times6/2) – 20\times3 = - 45kNm$

Point of zero shear at C.

$M_{max} = M_C = 30kNm$

Point of contra – flexure:

Since BM values change from –ve to +ve in the portion of point of contra – flexure is in the portion AC & BM values changes from +ve to –ve is in the portion CB, another point of contra – flexure is in the portion CB.

Let the BM be 0 at D at a distance x.

$M_D = 0$

$-45 + (40\times x) - 10\times x\times x/2 = 0$

$5x^2 - 40x + 45 = 0$

$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{+40 \pm \sqrt{(-40)^2 - 4\times 5\times 45}}{2\times 5}$

$= \frac{40 \pm \sqrt{1600 - 900}}{10}$

= 6.65m & 1.35m

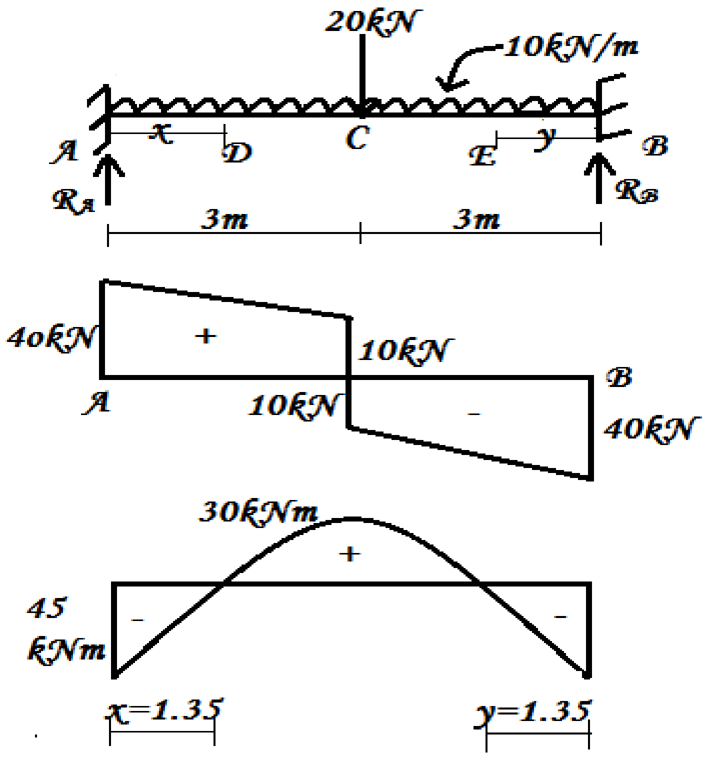
The points of contra flexure are x= 1.35m from A and also y = 1.35m from B.

x = y, D =E

∴ acceptable value 1.35m from A.

Since the loading is symmetrical.

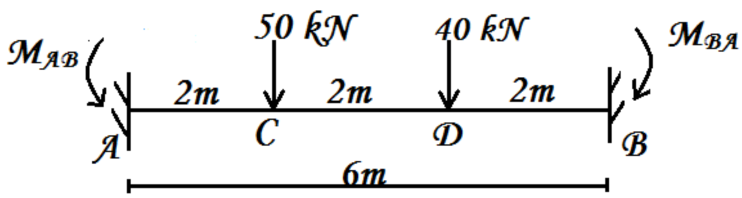
Y = 1.35 from B



Illustrative example for unsymmetrical loading

(1). A fixed beam of span 6m carries a point load of 50kN at a distance of 2m from the left end and another point load of 40kN at a distance of 2m from the right end. Calculate the fixed end moments and draw SFD & BMD. Locate the point of contra – flexure.

Given data:



To find:

1. Fixed end moments (M_{AB}, M_{BA})
2. Draw SFD & BMD
3. Locate point of contra – flexure

Solution:

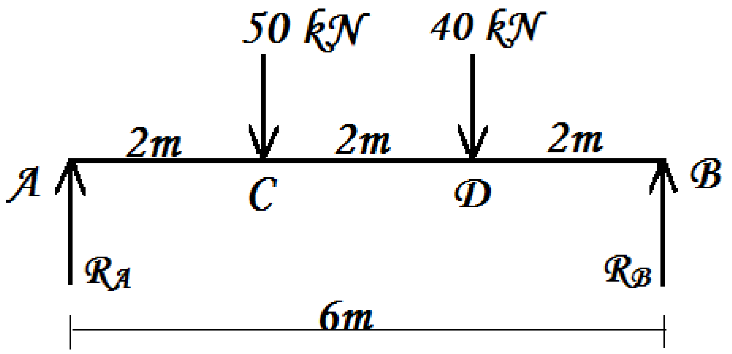
Though symmetry exists with respect to type of load and position, the values are not same.

This is case of unsymmetrical loading.

$\therefore M_{AB} \neq M_{BA}$

Step 1: (1) μ diagram

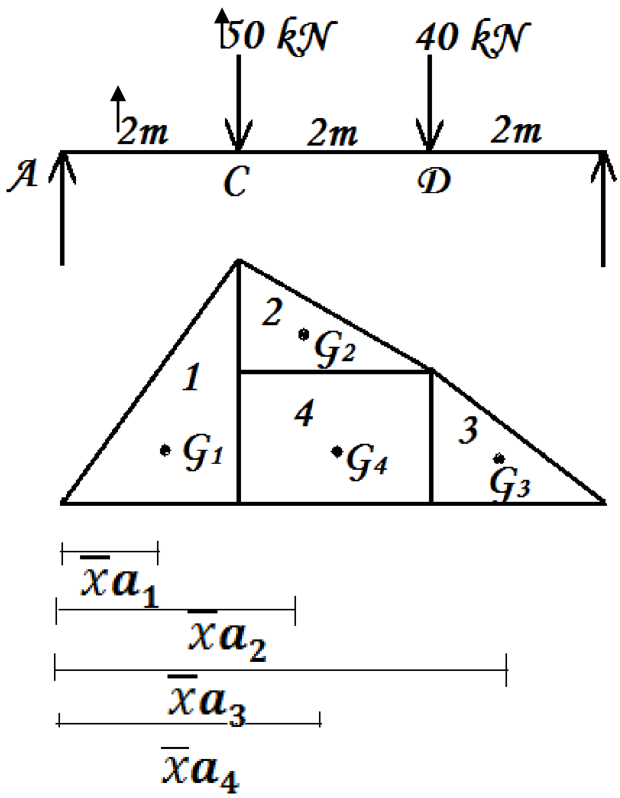
Considering the beam with loading as simply supported μ diagram



$R_A + R_B = 50 + 40$

$= 90kN$

Taking moments about A,



μ diagram

$M\mu_A = 0$ (simple support)
$M\mu_C = + R_A \times 2 = +46.67 \times 2$
$= + 93.34$kNm
$M\mu_D = + R_A \times 4 - (50 \times 2) = + 46.67 \times 4 - 100$
$= + 86.68$kNm
$M\mu_B = 0$
Complete μ diagram

(2) Centroidal distances:

$\bar{x}a_1 = \frac{2}{3} \times 2 = 4/3 \quad = 1.33$m from A
$\bar{x}a_2 = 2 + \frac{1}{3} \times 2 \quad = 2.66$m from A
$\bar{x}a_3 = 2 + 2 + \frac{1}{3} \times 2 \quad = 4.66$m from A
$\bar{x}a_4 = 2 + \frac{2}{2} \quad = 3$m from A

(3). Area of μ diagram

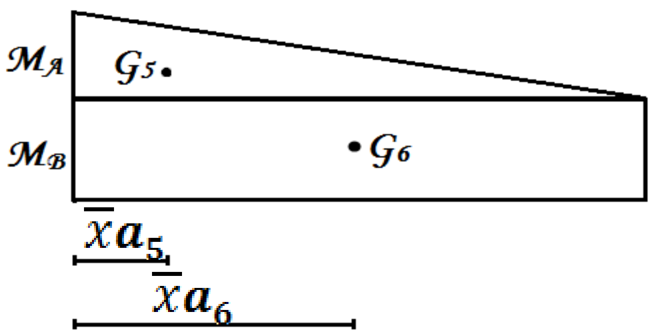
Area of (1) $a_1 = \frac{1}{2}bh$
$= \frac{1}{2} \times 2 \times 93.34$
$= 93.34\text{kNm}^2$
Area of(2) $a_2 = \frac{1}{2}bh$
$= \frac{1}{2} \times 2 \times (93.34 - 86.68)$
$= \frac{1}{2} \times 2 \times 6.66 = 6.66\text{kNm}^2$
Area of (3) $a_3 = \frac{1}{2}bh$
$= \frac{1}{2} \times 2 \times 86.68 = 86.68\text{kNm}^2$
Area of (4) $a_4 = l \times b$
$= 2 \times 86.68 = 173.36\text{kNm}^2$

∴ Area of μ diagram = Area of (1) + Area of (2)+ Area of (3) + Area of (4)
$= 93.34 + 6.66 + 86.68 + 173.36$
$= 360.04\ \text{kNm}^2$

(4). Moment of area of μ diagram about A:

$= \sum a . \bar{x}a$
$= a_1 . \bar{x}a_1 + a_2 . \bar{x}a_2 + a_3 . \bar{x}a_3 + a_4 . \bar{x}a_4$
$= (93.34 \times 1.33) + (6.66 \times 2.66) + (86.68 \times 4.66) + (173.36 \times 3)$
$= 124.14 + 17.72 + 403.93 + 520.08$
$= 1065.87\text{kNm}^3$.

Step 2: μ' diagram



Since the loading is unsymmetrical μ' diagram is a trapezium. Assuming $M_A > M_B$.

Centroidal distances:

$\bar{x}a_5 = \frac{1}{3} \times 6$ = 2m from A

$\bar{x}a_6 = l/2 = 6/2 = 3$ from A

Area of μ' diagram:

= Area of (5)a_5 + Area of (6)a_6

Area of (5) a_5 = ½ bh

$= ½ \times 6 \times (M_A - M_B)$

$= 3 \times (M_A - M_B)$

$= 3M_A - 3M_B$

Area of (6) $a_6 = l \times b$

$= 6 \times M_B$

$= 6M_B$

∴ Area of μ' diagram = $a_5 + a_6$

$= 3M_A - 3M_B + 6M_B$

$= 3M_A + 3M_B$

Moment of area of μ' diagram about A:

$= a_5 . \bar{x}a_5 + a_6 . \bar{x}a_6$

$a_5 . \bar{x}a_5 = 3(M_A - M_B) \times 2 = 6(M_A - M_B)$

$a_6 . \bar{x}a_6 = 6M_B \times 3 = 18 M_B$

$a_5 . \bar{x}a_5 + a_6 . \bar{x}a_6 = 6 \times (M_A - M_B) + 18 M_B$

$= 6M_A - 6M_B + 18 M_B$

$= 6M_A + 12M_B$

Step 3: Fixed end moments

Since the loading is unsymmetrical $M_{AB} \neq M_{BA}$

Applying Mohr's theorem I

Area of μ' diagram = Area of μ diagram

$3M_A + 3M_B = 360.64$

$M_A + M_B = 120.21$ - (1)

Applying Mohr's theorem II

Moment of area of μ' diagram about A = Moment of area of μ diagram about A

$6M_A + 12M_B = 1065.67$

$6 (M_A + 2M_B) = 1065.67$

$M_A + 2M_B = 177.61$ - (2)

$M_A + M_B = 120.21$ - (1)

Subtracting (1) from (2)

$M_B = 177.61 - 120.21$

∴M_B = 57.4kNm (hogging)

M_A = 62.81kNm (hogging)

Step 4: Vertical reactions (R_A, R_B)

Already calculated in step 1.

∴ R_A = 46.67kN R_B = 43.33kN

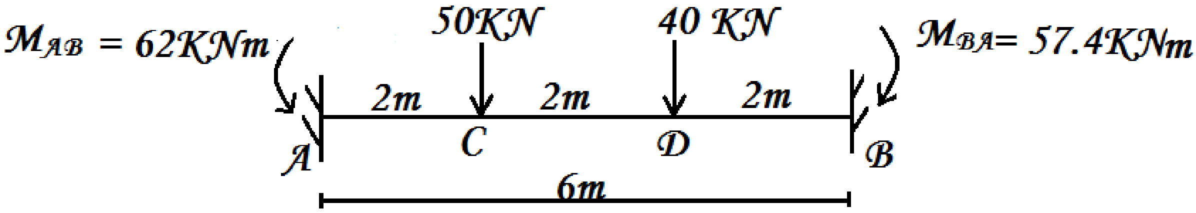
Step 5: Shear force (V_X)

V_A (L) = 0
V_A (R) = + R_A = + 46.67kN
V_C (L) = + V_A (R) = + 46.67kN
V_C (R) = + R_A – 50 = + 46.67 – 50
= - 3.33kN
V_D (L) = V_C (R) = - 3.33kN
V_D (R) = + R_A – 50 – 40
= +46.67 – 90 = - 43.33kN
V_B (L) = V_D (R) = - 43.33kN
V_B (R) = 0

Since S.F changes its sign exactly at C, +M_{max} occurs at C.

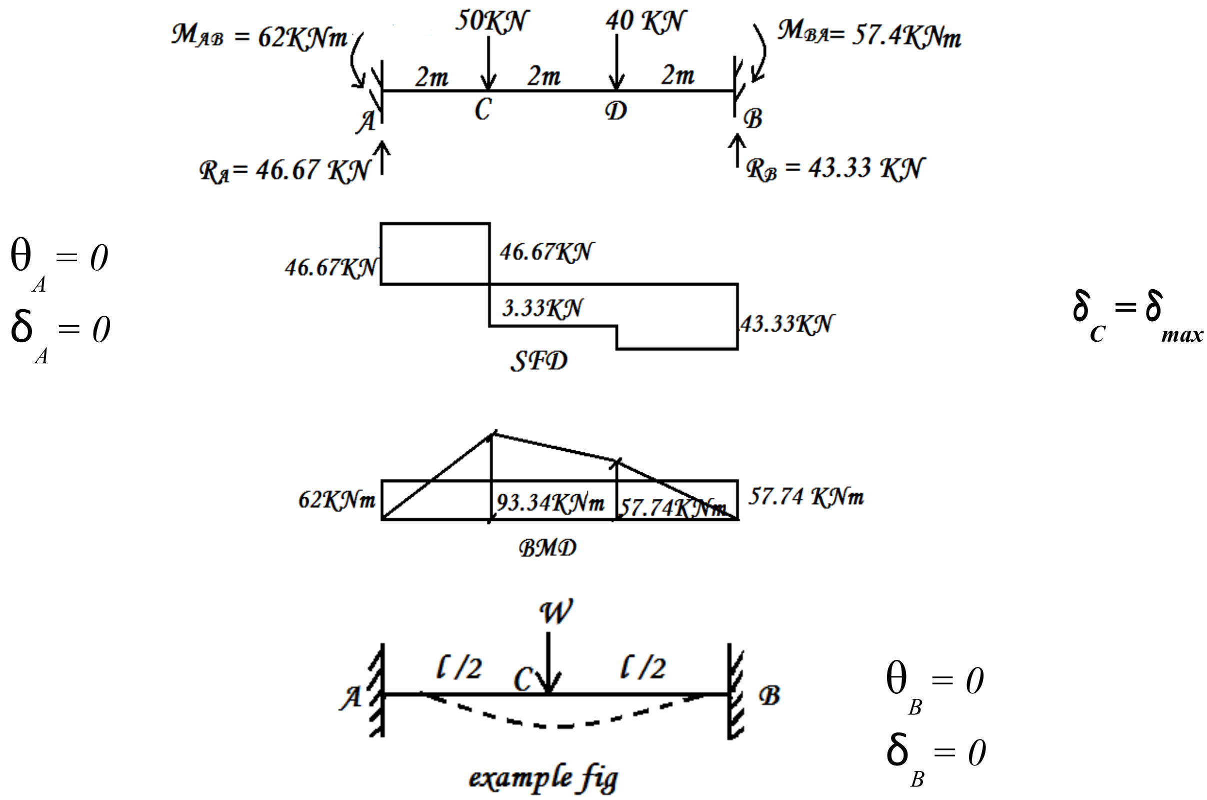
Step 6: Bending moment (M_X)

M_A = - M_{AB}
=-
62kNm



M_C = - M_{AB} + R_A × 2
= -62 + 46.67 × 2 = + 31.34kNm
M_D = - M_{AB} + R_A × 4 – 50 × 2
= -62 + 46.67 × 4 – 100 = + 24.68kNm
M_B = - M_{BA} = - 57.74kNm

Step 7: Summary



Slope and Deflection of symmetrically loaded Fixed beams by Area – moment method

In case of symmetrically loaded fixed beams, the slopes and deflections at the fixed supports are 0.

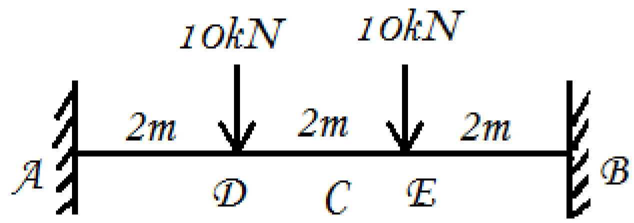
- In case of symmetrically loaded fixed beams, the max deflection occurs at mid span (Centre).
- For finding the maximum deflection, Mohr's theorem II is applied between A & C or B & C.
- Only half the BMD is considered.

WORKED EXAMPLE ON DEFLECTION FOR SYMMETRICALLY LOADED FIXED BEAMS.

(1) A fixed beam of span 6m carries concentrated load of 10kN at a distance of 2m from the left end and another concentrated load of 10kN at a distance of 2m from the right end. Calculate the max deflection by Area – moment method.

Given data:

$$\delta_C = \delta_{max}$$



To find:

The max deflection by Area – mom

Solution:

Mohr's theorem II is applied between the p

Step 1: Fixed end moments (M_{AB} = M_{BA} = M

For symmetrically loaded fixed beam M_{AB} =

∴ Applying Mohr's theorem I,

Area of μ diagram = Area of μ' diagram

Area of half μ diagram = $a_1 + a_2$

a_1 = ½ bh

= ½ × 2 × 20

= 20kNm^2

a_2 = l b

= 1 × 20

= 20kNm^2

Total area of μ diagram = 2 (20 + 20)

= 80kNm^2

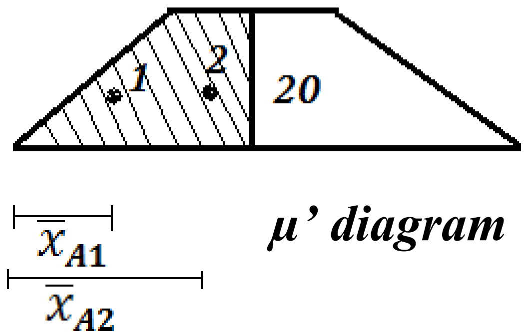
Area of μ' diagram (a_3) = l b

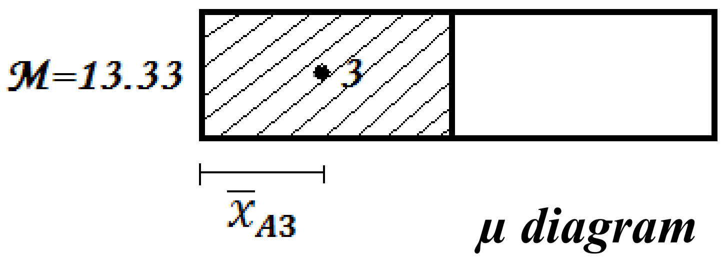
= 6 M

Area of μ' diagram = Area of μ diagra

6 M = 80

M = 80 / 6 = 13.33kNm.





Step 2: Max deflection (δ_{max}):

$$\delta_{max} = \delta_C$$

Applying Mohr's theorem II between A & C

$$\delta_{max} = \delta_C = \frac{\Sigma a_C \bar{x}_A}{EI}$$

$\Sigma a_C\, \bar{x}_A = a_1 .\bar{x}_{A1} + a_2 .\bar{x}_{A2} - a_3 .\bar{x}_{A3}$

$\bar{x}_{A1} = \frac{2}{3} \times 2 = 4 / 3$ = 1.33m from A

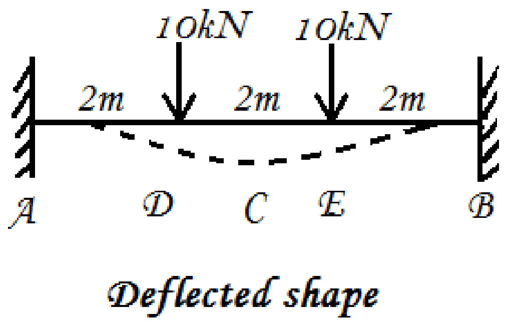
$\bar{x}_{A2}$ = 2 + 1 / 2 = 2.5m from A

$\bar{x}_{A3}$ = 3 / 2 = 1.5m from A

∴$\Sigma a_C\, .\, \bar{x}_A$ = 20 × 1.33 + 20 × 2.5 – 3 × 13.33 × 1.5

= 16.61

∴$\delta_{max} = \delta_C = \frac{\Sigma a_C \bar{x}_A}{EI} = \frac{16.61}{EI}$ m.



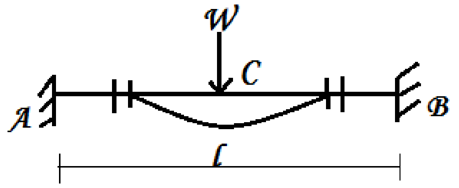
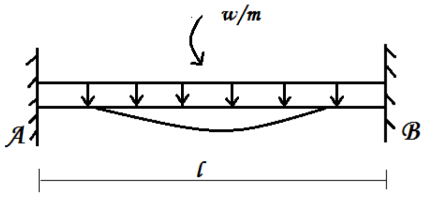
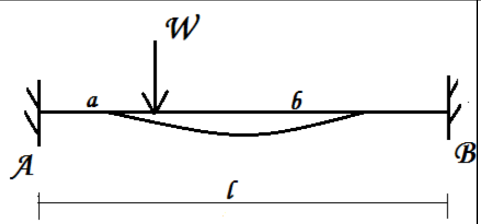
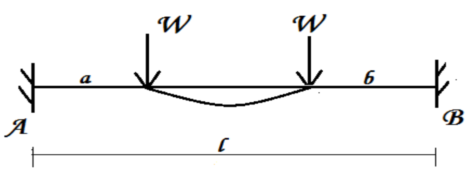
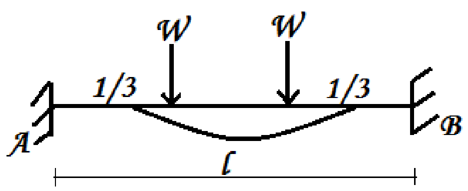
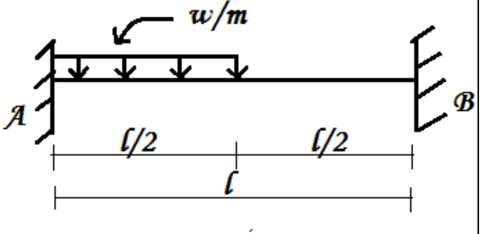
When EI is in kNm^2.

Result:

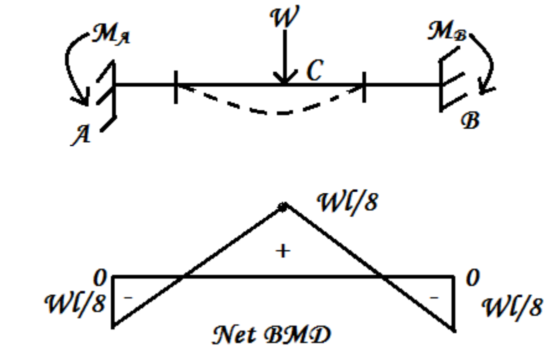
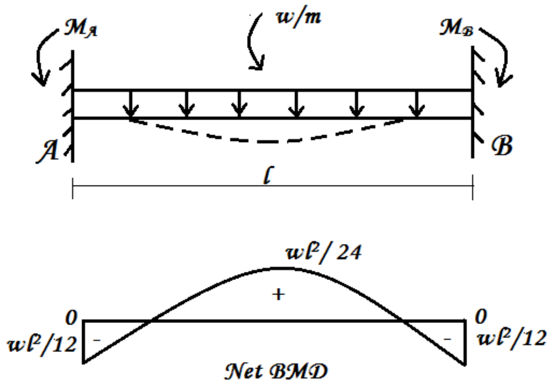
Maximum deflection = $\frac{16.61}{EI}$ m.

HIGH LIGHTS

1). Fixed beam:

Sl.no	Beam	M_A	M_B	δ_{max}
1.		$-\frac{Wl}{8}$	$-\frac{Wl}{8}$	$\frac{Wl^3}{192\ EI}$
2.		$-\frac{wl^2}{12}$	$-\frac{wl^8}{12}$	$\frac{wl^4}{384\ EI}$
3.		$-\frac{Wab^2}{l^2}$	$-\frac{Wa^2b}{l^2}$	$\frac{Wa^3b^3}{3\ EI\ l^3}$
4.		$-\frac{Wa(l-a)}{l}$	$-\frac{Wa(l-a)}{l}$	$\frac{Wa^2(3l-4a)}{24\ EI}$
5.		$-\frac{2}{9}\ W\ l$	$-\frac{2}{9}\ W\ l$	–
6.		$-\frac{11}{192}\ wl^2$	$-\frac{5}{192}\ wl^2$	–

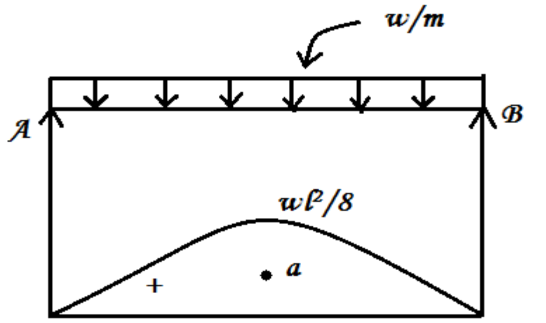
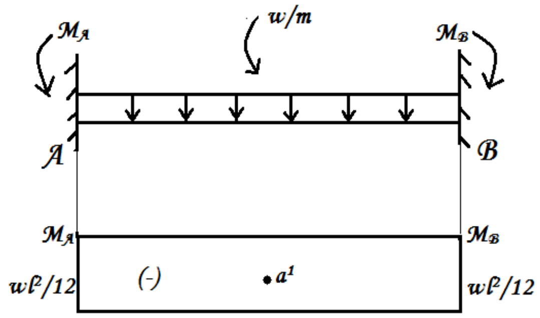
2). Net bending moment:

1)		$M_c = +\frac{Wl}{8}$
2)		$M_c = +\frac{wl^2}{24}$

3). Simply supported beam and Fixed beam:

Sl.no	Simply supported beam	Fixed beam
1.	Ends are simply supported	Ends are fixed
2.	Ends are free to rotation	Ends are restricted against rotation
		Slope at the ends are zero
3.	Slope at the ends are maximum	
	Moment at the ends are zero	Moment at the end is maximum
4.		Moment at the centre is much reduced
	Moment at the centre is maximum	Deflection at the centre is less
5.	Deflection at the centre is more	
6.	It is a determinate beam.	It is a indeterminate beam.
7.		

4). Free BMD & Fixed BMD:

Sl. no	Free BMD	Fixed BMD
1)	The bending moment diagram drawn considering the beam as a simply supported beam is called free bending moment diagram.	The bending moment diagram drawn only for the fixed end moments are called fixed bending moment diagram.
2)	It is denoted by μ diagram	It is denoted by μ' diagram.
3)	It is sagging bending moment	It is hogging bending moment.
	 Free BMD	 Fixed BMD

QUESTIONS

Two mark Questions:

1) Draw the bending moment diagram for the fixed beam carrying UDL throughout.
2) What is the fixed beam and How is differ from Simply supported beam?
3) What will be slope at the fixed end of the fixed beam carrying UDL throughout its length?
4) Draw the BMD for a fixed beam subjected to a point load at the mid-span.
5) State the maximum deflection value in a fixed beam subjected to a UDL throughout the span.
6) State any two advantages of a fixed beam.
7) Write any one advantage of a fixed beam compared to simply supported beam.
8) Define free BMD.
9) Show that the area of free BMD and fixed BMD in a fixed beam are equal.
10) Define Free BMD? (unit 2.1)

Three mark Questions:

1) Calculate fixed end moment and maximum deflection in a fixed beam of span 5m subjected to a central point load of 30kN. Take EI = 1.20×10^4 kNm^2.
2) A fixed beam of 6m span subjected to a UDL of w/m over its full length. The net BM at the centre is 30kN/m. find the value of w.
3) Show that the area of BMD and fixed BMD in a fixed beam are equal.
4) State the different method of Analysis of Indeterminate structures.

Ten mark Questions:

1) A fixed beam of span 9m is subjected to an UDL of 20kN/m over the entire length. It is also carries two concentrated loads of 10kN each at 3m from the ends.

i. Determine the values of fixing moments.

ii. Sketch the BMD marking the maximum values there in.

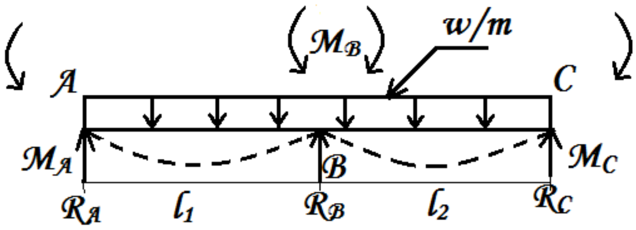
2) A fixed beam of 12m span carries two point loads of 60kN and 30kN at distance of 3m & 6m from left end support respectively. Draw the SF and BM diagrams, using area moment method.
3) A fixed beam of span 8m is subjected to an UDL of 4kN/m over a length of 4m. Symmetrically placed at centre portion. Determine the support moments and draw the BMD.
4) A fixed beam of span 'l' carries a non-centric concentrated load of 'w' at distance 'a' from the left support and 'b' from the right support. Derive the expression for the fixed end moments using Mohr's Theorems (Area-moment method).
5) A fixed beam of span 5m carries two equal point loads of 20kN each at 2m from each end. Find the fixed end moments. Draw the SFD and BMD.
6) A fixed beam of span 6m carries a central point load of 20kN in addition to an UDL of 10kN/m over the entire span. Calculate the fixed end moments. Draw the BMD.
7) A fixed beam of span 6m carries a central point load of 30kN and 50kN at 2m and 4m from the left support respectively. Find the support moments and draw SFD and BMD.
8) A fixed beam of span 5m carries a central point load of 16kN. Determine the fixing moments and draw SFD and BMD. Find the maximum central deflection.
9) A fixed beam of 6m span subjected to a two concentrated load of 30kN at a distance of 2m from both ends. Draw SFD and BMD.
10) A fixed beam of span 8m carries an UDL of 45kN/m over the entire span. It also carries two point loads of 150kN each at 2m from the ends. Calculate the support moments. Draw the BMD.

2.2. CONTINUOUS BEAMS

THEOREM OF THREE MOMENTS METHOD

Introduction to Continuous beams:

When the beam has more than two supports, it is called as **continuous beam**. Hogging moment will be developed at the intermediate supports. Hence, it is stiffer and stronger than other beams.
It is statically in determinate beam. The slope and deflection are less. It can carry more loads than other type of beams. Continuous beams are economical.



Fig

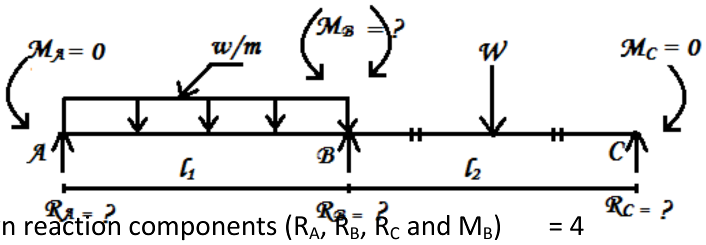
Degree of indeterminacy of continuous beam:

The difference between number of known reaction components and number of known static equation is called Degree of indeterminacy.
i.e. D.I. = (No of known reaction components – No of known static equation)

The degree of indeterminacy is depend upon the end conditions, no of spans and type of supports as given below.

a. Two span continuous beam:

1. **Continuous beam with both end simply supported:**

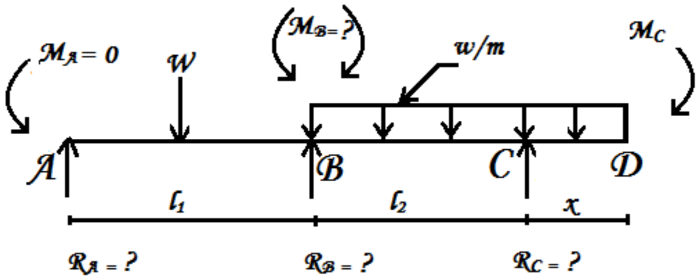


No of unknown reaction components (R_A, R_B, R_C and M_B) = 4

No of available static equilibrium Equations ($\sum_V = 0$, $\sum_M = 0$) = 2

Degree of indeterminacy = (4 – 2) = 2

2. **Continuous beam with one end simply supported and other end with overhanging:**

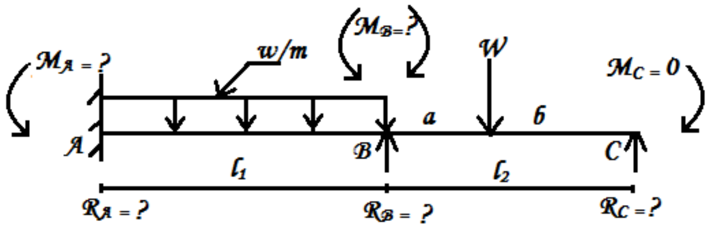


No. of unknown reaction components (R_A, R_B, R_C & M_B) = 4

No. of available static equilibrium eqns ($\sum_V = 0$, $\sum_M = 0$) = 2

Degree of indeterminacy = (4 – 2) = 2

3. Continuous beam with one end fixed and other end simply supported:

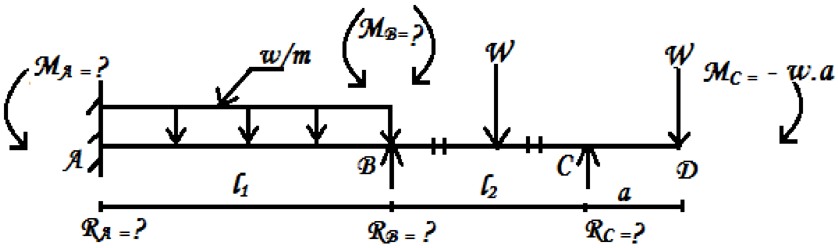


No. of unknown reaction components (R_A, R_B, R_C & M_A, M_B) = 5

No. of available static equilibrium eqns ($\sum_V = 0$, $\sum_M = 0$) = 2

Degree of indeterminacy = (5 – 2) = 3

4. Continuous beam with one end fixed and other end with over hanging:

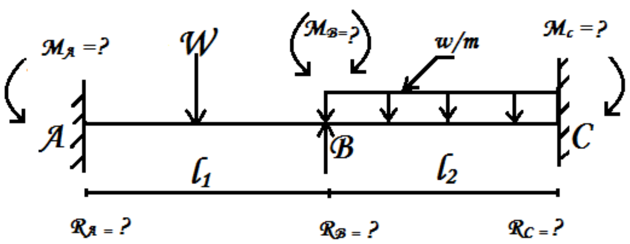


No. of unknown reaction components (R_A, R_B, R_C & M_A, M_B) = 5

No. of available static equilibrium eqns ($\sum_V = 0$, $\sum_M = 0$) = 2

Degree of indeterminacy = (5 – 2) = 3

5. Continuous beam with both ends fixed:



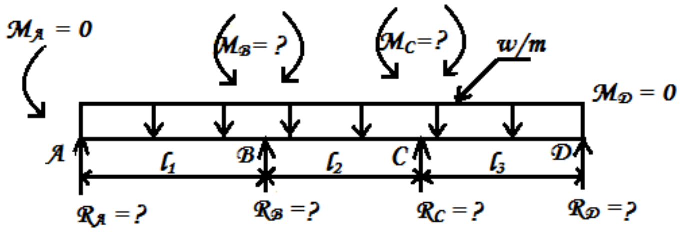
No. of unknown reaction components (R_A, R_B, R_C, M_A, M_B & M_C) = 6

No. of available static equilibrium eqns ($\sum_V = 0$, $\sum_M = 0$) = 2

Degree of indeterminacy = (6 – 2) = 4

b. Three span continuous beam:

1) Both ends are simply supported:



No. of unknown reaction components (R_A, R_B, R_C, R_D, M_B & M_C) = 6

No. of available static equilibrium eqns ($\sum_V = 0$, $\sum_M = 0$) = 2

Degree of indeterminacy = (6 – 2) = 4

General methods of analysis of continuous beam:

Method of analysis of continuous beam.

A. Theorem of three moments method
B. Moment distribution method
C. Area moment method
D. Slope and deflection method
E. Strain energy method
F. Khani's rotation method
G. Column analogy method
H. Influence line method

Theorem of Three Moment's Method (or) Clapeyron's Theorem of Three Moment Method:

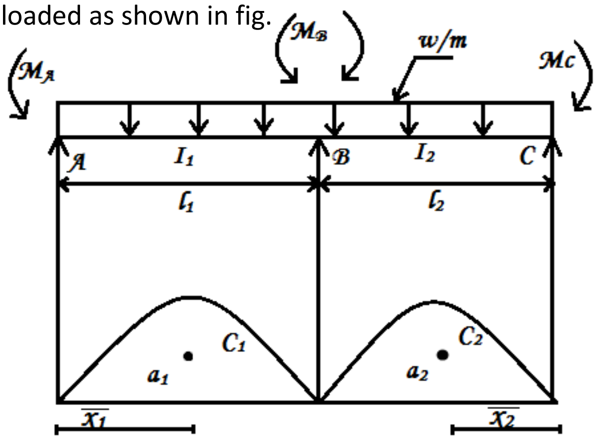
In this method Mohr's theorem I & II are used for determinate the support moment of a continuous beam. It is based on the geometric condition of the deformed continuous beam.

This method was introduced by Engineer Clapeyron's using three support moments. Hence, it is called Clapeyron's theorem of three moment's method.

Statement of Clapeyron's Theorem of three moments:

Clapeyron's theorem states that if a beam has 'n' supports, the end being fixed than the same number of equations required to determining the support moments may be obtained from the consecutive pairs of spans i.e. AB – BC, BC – CD, CD – DE and so on.

Continuous beam ABC loaded as shown in fig.



Let,

M_A, M_B, M_C are the support moments.

$M_A \left(\frac{l_1}{I_1}\right)$ + 2 $M_B \left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right)$ = - $\left[\frac{6a_1\bar{x}_1}{l_1 I_1} + \frac{6a_2\bar{x}_2}{l_2 I_2}\right]$

Theorem of three moments equation

When,

$l_1 = l_2 = l$

$$\mathbf{M_A, l_1 + 2\, M_B\, (l_1 + l_2) + M_C\, l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right]}$$

Where,

a_1 = area of free BMD for span AB
a_2 = area of fixed BMD for span BC
$\bar{x}_1$ = c.g of BMD from left end (A)
$\bar{x}_2$ = c.g of BMD from right end (C)

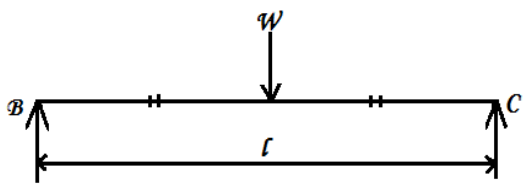
<u>Application of Clapeyron's Theorem of three moments:</u>

The following solved problems are the examples for the Application of Clapeyron's Theorem of three moments.

<u>Area of free BMD for standard cases:</u>

1) Simply supported beam with central point load:

Max B.M = $\frac{Wl}{4}$

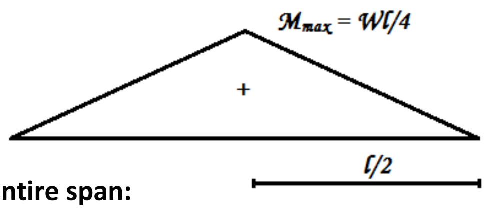


Area, a = ½ × l × $\frac{Wl}{4} = \frac{Wl^2}{8}$

$\bar{x}$ = $\frac{l}{2}$ centroid from supports.

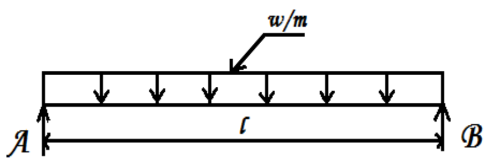
$\therefore \frac{6a\bar{x}}{l} = \frac{6\times\left[\frac{Wl^2}{8}\right]\times\frac{l}{2}}{l} = \frac{3}{8}\, Wl^2$

$$\mathbf{\frac{6a\bar{x}}{l} = \frac{3}{8}\, Wl^2}$$



2) Simply supported beam with UDL of w/m over its entire span:

Max B.M = $\frac{wl^2}{8}$

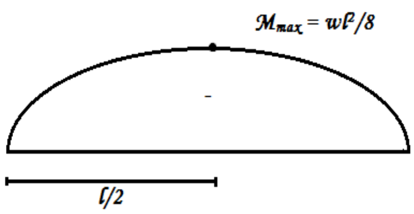


Area a = $\frac{2}{3} \times l \times \frac{wl^2}{8} = \frac{wl^3}{12}$

$\bar{x}$ = $\frac{l}{2}$ centroid from either supports

$\therefore \frac{6a\bar{x}}{l} = \frac{6\times\left[\frac{Wl^3}{12}\right]\times\frac{l}{2}}{l} = \frac{wl^3}{4}$

$$\mathbf{\frac{6a\bar{x}}{l} = \frac{wl^3}{4}}$$



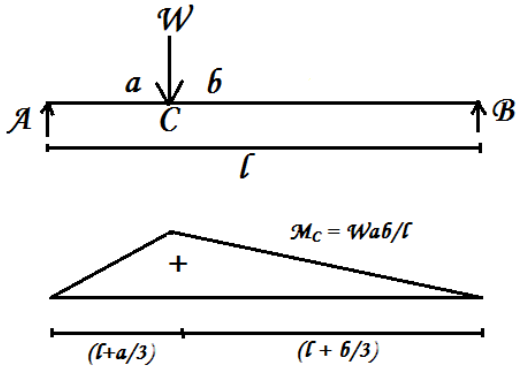
3) **Simply supported beam with non – central load (W):**

Max B.M = $\frac{Wab}{l}$

Area a = ½ × l × $\frac{Wab}{l}$; **a =** $\frac{Wab}{2}$

Centroid from left end $\overline{x_1} = \left[\frac{l+a}{3}\right]$

Centroid from right end $\overline{x_2} = \left[\frac{l+b}{3}\right]$



$\therefore \frac{6a\overline{x_1}}{l}$ from left end A = $\frac{6 \times \left[\frac{Wab}{2}\right] \times \left[\frac{l+a}{3}\right]}{l} = \frac{Wab(l+a)}{l}$

$\frac{6a\overline{x_1}}{l} = \frac{Wa(l-a)(l+a)}{l} = \frac{Wa(l^2-a^2)}{l}$

$\frac{6a\overline{x_1}}{l} = \frac{Wa(l^2-a^2)}{l}$ from left end

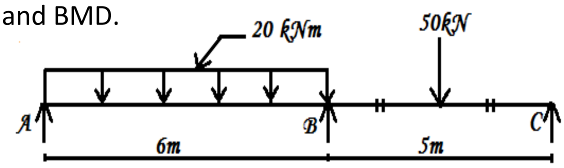
III ly $\frac{6a\overline{x_2}}{l} = \frac{Wb(l^2-b^2)}{l}$ from right end

<u>TWO SPAN CONTINUOUS BEAM</u>

<u>TYPE – 1 Both Ends Are Simply Supported</u>

Problem 1:

A Continuous beam ABC is Simply Supported at A and C such that AB = 6m and BC = 5m. The span AB carries an UDL of 20kN/m and the span BC carries a point load of 50kN at the centre. Find the support moments by using theorem of three moments draw SFD and BMD.

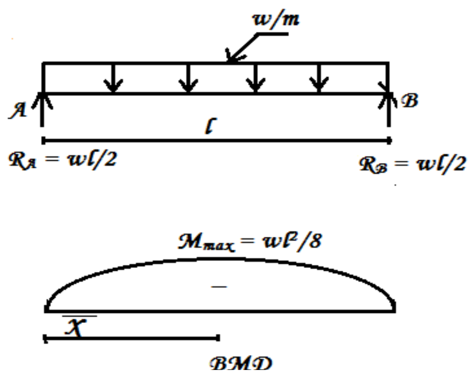


<u>**Solution**</u>:

1) <u>**Simply Supported beam moments:**</u>

Considering a each span as Simply

Supported beam and draw free BMD

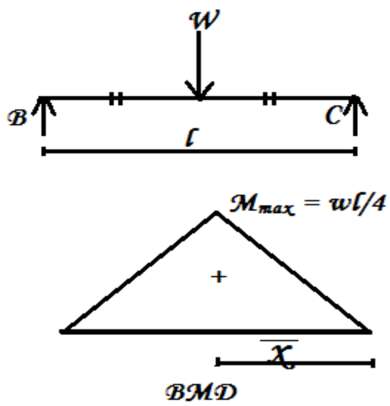


Span AB:

M = $\frac{wl^2}{8} = \frac{20 \times 6^2}{8}$ = 90kNm

Span BC:

M = $\frac{wl}{4} = \frac{50 \times 5}{4}$ = 62.5kNm



Using theorem of three moments method.

2) <u>**Support moments**</u>:

Span AB & BC

AB = l_1 = 6m; BC = l_2 = 5m

Since end A & C are simply supported

$M_A = M_C = 0$

Applying theorem of three moment equation

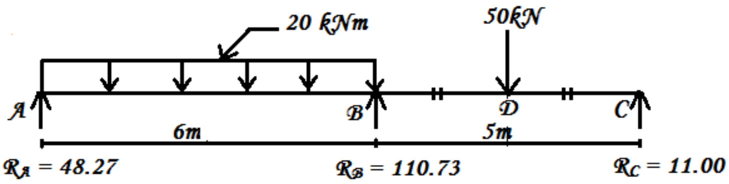
$M_A l_1 + 2M_B\,(l_1 + l_2) + M_C\,l_2 = -\,6\left[\frac{a_1\overline{x_1}}{l_1} + \frac{a_2\overline{x_2}}{l_2}\right]$

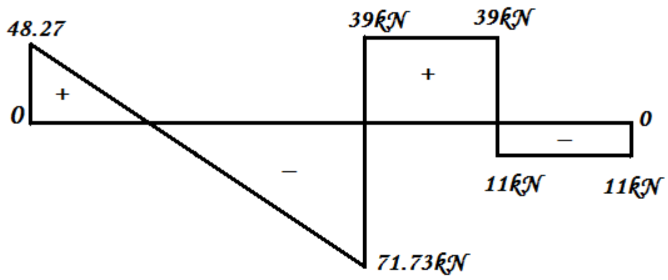
$$0 + 2M_B\,(6+5) + 0 = -\,6\left[\frac{\left(\frac{2}{3}\times6\times90\right)\left(\frac{6}{2}\right)}{6} + \frac{\left(\frac{1}{2}\times5\times62.5\right)\left(\frac{5}{2}\right)}{5}\right]$$

$22\ M_B = -1548.75$

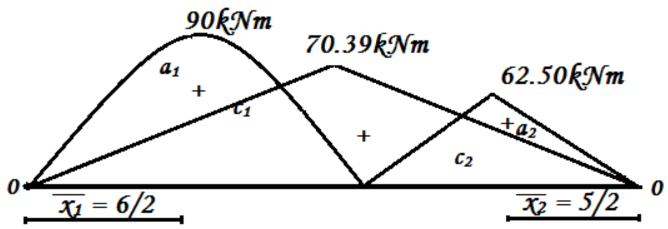
$M_B = 70.40$kNm.

Draw SFD and BMD:





SFD



BMD

Final support moments:

$M_A = 0$

$M_B = -\ 70.40$kNm

$M_C = 0$

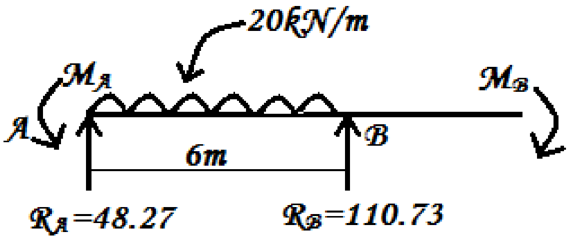
Reactions:

support	A	B		C
Reaction due to free BM $w\frac{l}{2}, \frac{W}{2}$	+60	+60	+25	+25
Reactions due to fixing moments $\frac{M_A \sim M_B}{l_1}, \frac{M_B \sim M_C}{l_2}$	-11.73	+11.73	+14	-14
Net reactions		71.72	39	
	R_A = 48.27kN	R_B=110.73kN		R_C = 11kN

Alternate method:

For span AB

Taking moment about 'B'



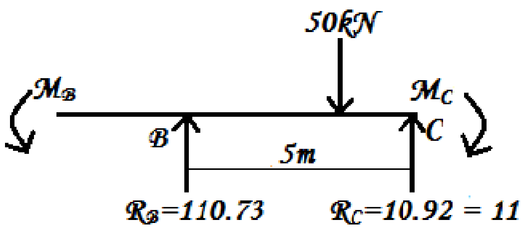
$M_A + w.l.\frac{l}{2} = R_A \times l + M_B$

$0 + 20 \times 6 \times \frac{6}{2} = R_A \times 6 + 70.40$

$R_A = 48.27$kN.

For span BC

Taking moment about 'B'



$M_B + R_C \times l = W\frac{l}{2} + M_C$

$70.40 + R_C \times 5 = 50 \times \frac{5}{2} + 0$

$R_C = 11$kN

R_B = Total load – ($R_A + R_B$) = (20×6×50) – (48.27 + 11)

$R_B = 110.73$kN

Net BM:

Max BM will occur at x distance from A,

71.73 x = (6-x) (48.27)

= 289.62 – 48.27x

120 x = 289.62

$x = \frac{289.62}{120}$

$x = 2.40m$

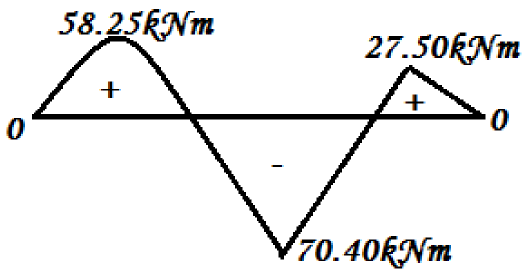
$M_A = 0$

$M_x = 48.27 \times 2.40 - 20 \times 2.40 \times \frac{2.40}{2} = 58.25kNm$

$M_B = -70.40kNm$

$M_D = R_C \times \frac{5}{2} = 11 \times \frac{5}{2} = 27.50kNm$

Draw BMD



Net BMD

Problem 2:

A continuous beam ABC of length 8m has two equal spans. The span AB carries an UDL of 20kN/m over its entire length and the span carries a point load of 20kN at 3m from B. Draw SFD and BMD. Take ends A & C are simply supported. Apply the theorem of three moments.

Solution:

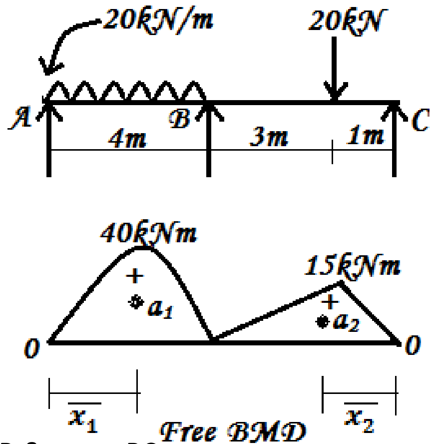
i) **Draw free BMD for each span**

For span AB

$M_{AB} = \frac{wl^2}{8} = \frac{20 \times 4^2}{8} = 40kNm$

For span BC

$M_{BC} = \frac{Wab}{l} = \frac{20 \times 3 \times 1}{4} = 15kNm$



Free BMD

ii) **Support moments**:

Applying theorem of three moment equation for span AB & BC

$$M_A l_1 + 2M_B (l_1+l_2) + M_C l_2 = -6\left[\frac{a_1\bar{x}_1}{l_1} + \frac{a_2\bar{x}_2}{l_2}\right]$$

(Or)

$$M_A l + 2M_B (l_1+l_2) + M_C l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right] \quad \rightarrow (1)$$

Since the ends A and C are simply supported.

$$M_A = M_C = 0.$$

Using formula for standard cases:

i. Simply supported beam with UDL over entire span, $\frac{6a\bar{x}}{l} = \frac{wl^3}{4}$

$\therefore \frac{6a_1\bar{x}_1}{l_1} = \frac{20\times4^3}{4} = 320$

ii)simply supported beam with non-central load

$\frac{6a\bar{x}}{l} = \frac{Wb(l^2-b^2)}{l}$ from right end

$\therefore \frac{6a_2\bar{x}_2}{l_2} = \frac{20\times1(4^2-1^2)}{4} = 80$

Substituting in equation (1)

$M_A\times4 + 2M_B\,(4+4) + M_C\times4 = -\,[320+80]$

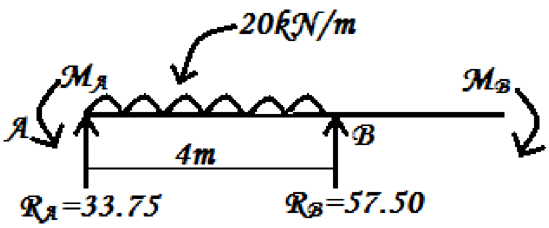
$0 + 16M_B + 0 = -\,400$

$M_B = \frac{-400}{16} = -\,25kNm$

Reactions:

Consider span AB

Taking moment at B



$=$

$M_A + \frac{wl^2}{l} = R_A \times l + M_B$

$0 + \frac{20\times4^2}{2} = 4\,R_A + 25$

$R_A = \frac{160-25}{4} = 33.75kN$

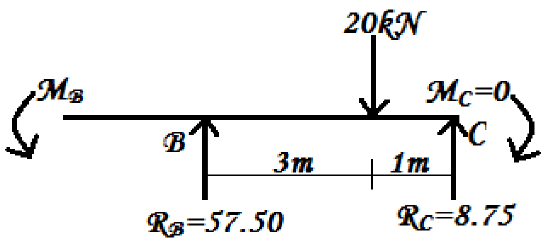
Consider span BC

Taking moment about B

$R_C \times l + M_B = M_C + (W \times a)$

$R_C \times 4 + 25 = 0 + (20 \times 3)$

$R_C = \frac{60-25}{4} = 8.75kN$



$\sum_V = 0$

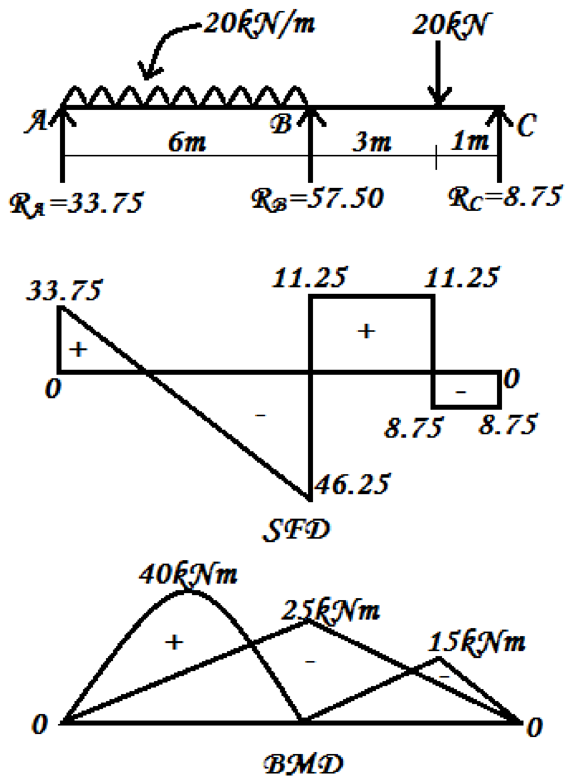
$R_A+R_B+R_C =$ Total load

$R_A =$ Total load $-\,(R_A + R_B)$

$R_B = (20\times4) + 20 - (33.75 + 8.75)$

$R_B = 57.50kN.$

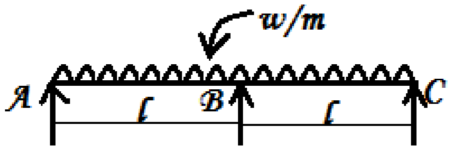
Draw SFD & BMD



Problem 3:

A two span continuous beam ABC of uniform flexural rigidity is supported to UDL of w/m. The two spans AB = BC = l. Determine the support moments by using Clapeyron's Theorem of three moments method. Ends A & C are simply supported Draw BMD.

Solution:



Span AB = BC = l

I. **Free Bending Moment Diagram:**

B.M for span AB: $M_{AB} = \frac{wl^2}{8}$ kNm

B.M for span BC: $M_{BC} = \frac{wl^2}{8}$ kNm

II. **Support Moments:**

Ends A & C are simply supported:

$M_A = M_C = 0$

Applying theorem of three moments

Spans AB & BC

$AB = l_1 = l$; $BC = l_2 = l$

Applying Clapeyron's three moments equation:

$$M_A l_1 + 2M_B (l_1+l_2) + M_C l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right]$$

Since spans AB = BC = l and carries equal UDL of w/m

$$a_1 = a_2 = \frac{2}{3} \times l \times \frac{wl^2}{8} = \frac{wl^3}{12}$$

$$\overline{x_1} = \frac{l}{2} \text{ from A; } \quad \overline{x_2} = \frac{l}{2} \text{ from C}$$

$$\frac{6a_1\overline{x_1}}{l_1} = \frac{6 \times \frac{wl^3}{12} \times \frac{l}{2}}{l_1} = \frac{wl^3}{4}$$

$$\text{III}^{ly} \; \frac{6a_2\overline{x_2}}{l_2} = \frac{wl^3}{4}$$

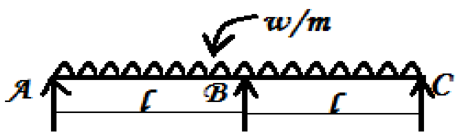
$$M_A\, l_1 + 2M_B\,(l_1+l_2) + M_C\, l_2 = -\left[\frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}\right]$$

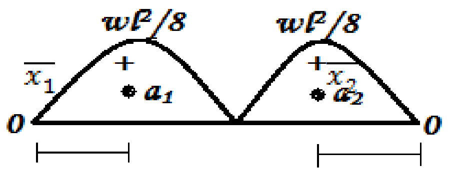
$$0 + 2\, M_B\,(l+l) + 0 = -\frac{wl^3}{4} + \frac{wl^3}{4};$$

$$0 + 4\, M_B\,.\, l + 0 = -\frac{wl^3}{2}$$

$$M_B = -\frac{wl^3}{2} \times \frac{1}{4l} = -\frac{wl^2}{8}$$

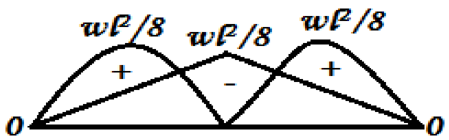
Draw BMD as shown in fig.





Free BMD

Support BMD



BMD

Results:

Final moments

$M_A = 0$

$M_B = -\frac{wl^2}{8}$

$M_C = 0$

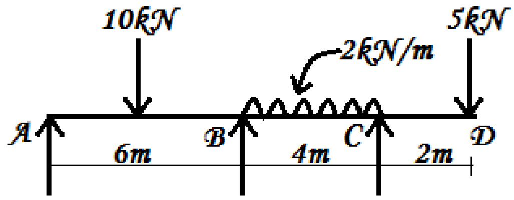
Type II One End Simply Supported

and Other End With Over Hanging

Problem 4:

A continuous beam ABCD is 12m long is simply supported at A, B & C span AB is 6m long and carries a point load of 10kN at its centre, span BC = 4m. Carries UDL of 2kN/m to its full length. Span CD carries a point load 5kN at free end D. Draw BMD by using theorem of three moments.

Solution:



i. **Draw free BMD for each span:**

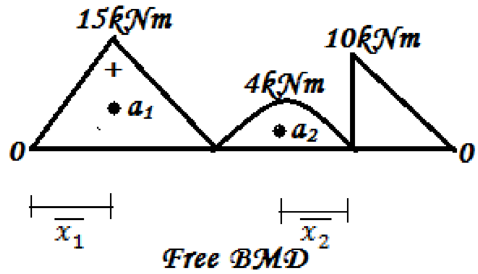
Span AB

$M_{AB} = \frac{Wl}{4} = \frac{10\times6}{4} = 15kNm$

Span BC

$M_{BC} = \frac{wl^2}{8} = \frac{2\times4^2}{8} = 4kNm$

$M_C = -W.a = -(5\times2) = -10kNm$



Free BMD

ii. **Support moments:**

Applying theorem of three moments equation

Span AB & BC

$$M_A l_1 + 2M_B (l_1+l_2) + M_C l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right] \rightarrow (1)$$

End A is simply supported

$\therefore M_A = 0$

Supported C is with over hanging

$\therefore M_C = -5 \times 2 = -10kNm$

Using formula for standard cases

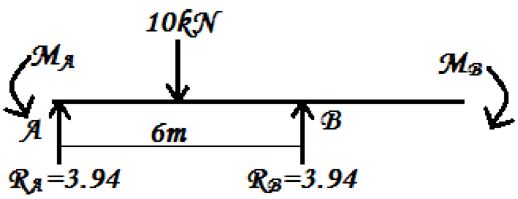
1. Simply supported with central point load, $\frac{6a\bar{x}_1}{l} = \frac{3Wl^2}{8}$

$\therefore \frac{6a_1\bar{x}_1}{l_1} = \frac{3\times10\times6^2}{8} = 135$

2. Simply supported with UDL, $\frac{6a\bar{x}}{l} = \frac{wl^3}{4}$

$\therefore \frac{6a_2\bar{x}_2}{l_2} = \frac{2\times4^3}{4} = 32$

Substituting in equation (1)



$M_A\times6 + 2M_B(6+4) + M_C\times4 = -[135 + 32]$

$0 + 20M_B + (-)10 \times 4 = -167$

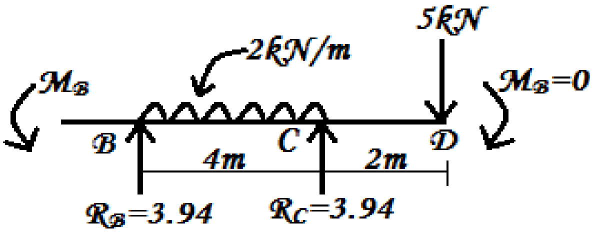
$20M_B = -167 + 40 = -127$

$M_B = \frac{-127}{20} = -6.35kNm$

Reaction:

Consider span AB

Taking moment about B



$$M_A + \frac{wl}{2} = R_A \times l + M_B$$

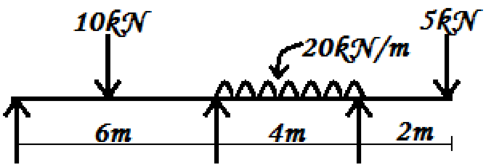
$$0 + \frac{10 \times 6}{2} = R_A \times 6 + 6.35$$

$$6\ R_A = (30 - 6.35) = 23.65\text{kNm}$$

$$R_A = \frac{23.65}{6} = 3.94\text{kNm}$$

Consider span BCD

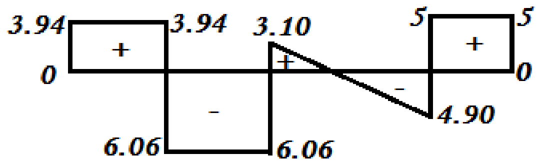
Taking moment about B



$$R_C \times 4 + M_B = (2 \times 4 \times \frac{4}{2}) + 5 \times 6$$

$$4\ R_C + 6.35 = 16 + 30 = 46$$

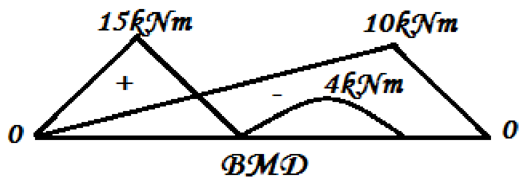
$$R_C = \left(\frac{6 - 6.35}{4}\right) = 9.90\text{kN}$$



$$\sum_V = 0$$

$$R_A + R_B + R_C = \text{Total load}$$

$$R_B = \text{Total loads} - (R_A + R_C)$$



$$R_B = (10 + (2 \times 4) + 5)$$

$$- (3.94 + 9.90)$$

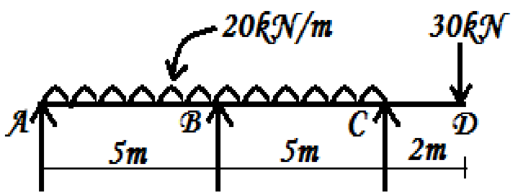
$$R_B = 9.16\text{kN}.$$

Draw SFD & BMD:

Problem 5:

A continuous beam of ABCD of length 12m is simply supported by three supports at A, B and C with an equal spacing of 5m. It carries an UDL of intensity 20kN/m over the two spans. There is a 30kN load on the free end, D. Analyse the beam using Clapeyron's Theorem and draw SFD & BMD.

Solution:

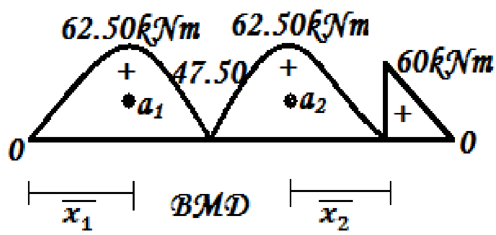


1. **Free BMD for each span:**

Span AB & BC

$$M_{AB} = M_{BC} = \frac{wl^2}{8}$$

$$= \frac{20 \times 5^2}{8} = 62.50\text{kNm}$$



Moment at C

$$M_C = -30 \times 2 = -60\text{kNm}$$

2. Support moments:

Applying Theorem of three moment equation for span AB & BC

$$M_A\, l_1 + 2M_B\,(l_1+l_2) + M_C\, l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right] \quad \rightarrow (1)$$

Since end A is simply supported, $M_A = 0$

Support 'C' is with over hanging $M_C = -30\times2 = -60$kNm

$$l_1 = \quad l_2 \quad = 5\text{m}$$

For standard cases

a) Simply supported with UDL, $\frac{6a\bar{x}}{l} = \frac{wl^3}{4}$

$$\frac{6a_1\bar{x}_1}{l} = \frac{6a_2\bar{x}_2}{l} = \frac{20\times6^3}{4} = 625$$

Substituting in equation (1)

$$M_A\times5 + 2M_B\,(5+5) + (-60\times5) = -\,[625 + 625]$$

$$0 + 20M_B - 300 = -\,1250$$

$$M_B = \frac{-1250+300}{20} = -\,4750\text{kNm}$$

Reactions:

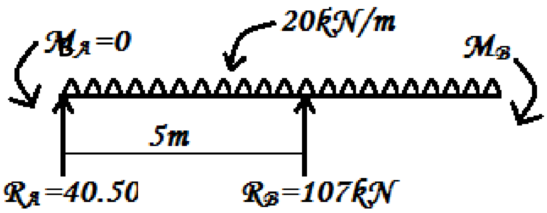
Consider span AB taking moment about B

$$M_A + \frac{wl}{2} = R_A\times l \times M_B$$

$$0 + \frac{20\times5^2}{2} = 5\,R_A + 47.50$$

A

$$R_A = \frac{250-47.50}{5} = 40.50\text{kN}$$



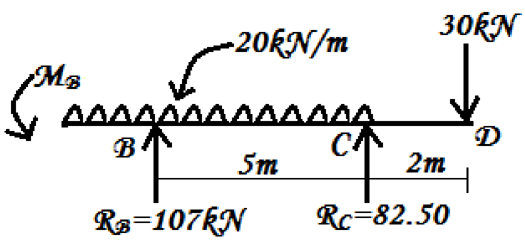
Consider span BC:

Taking moment about B

$$R_C\times1 + M_B = \left(\frac{20\times5\times5}{2}\right) + (30\times7)$$

$$R_C\times5 + 47.50 = (250+210) = 460$$

$$R_C = \left(\frac{460-47.50}{5}\right) \qquad = 82.50\text{kNm}$$



$$\sum V = 0$$

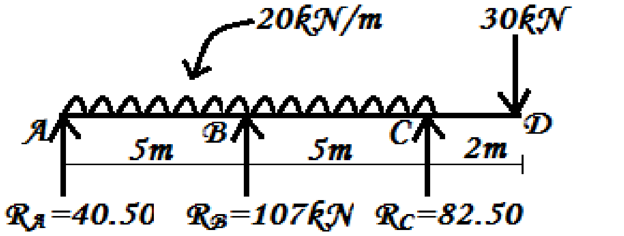
$$R_A + R_B + R_C = \text{Total load}$$

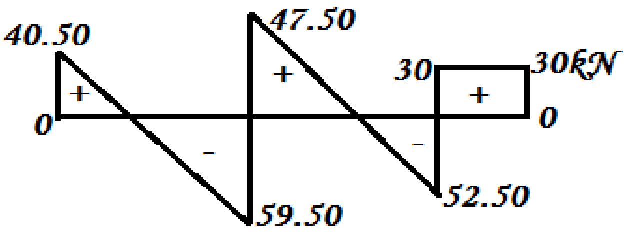
$$R_B = \text{Total load} - (R_A+R_C)$$

$$R_B = (20\times10) + 30 - (40.50+82.50)$$

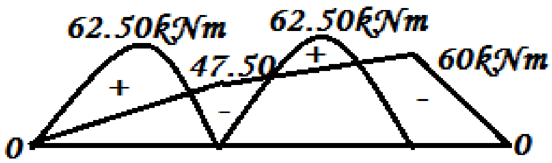
$$R_B = 107\text{kN.}$$

Draw SFD and BMD:





SFD



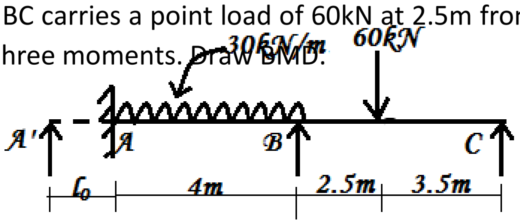
BMD

Type III One End Fixed And Other End Simply Supported

Problem 6:

A continuous beam ABC of span 10m is fixed at end A and simply supported at C span AB is 4m long and carries an UDL of 30kN/m over entire span and span BC carries a point load of 60kN at 2.5m from B. Determine the support moments by using theorem of three moments. Draw BMD.

Solution:



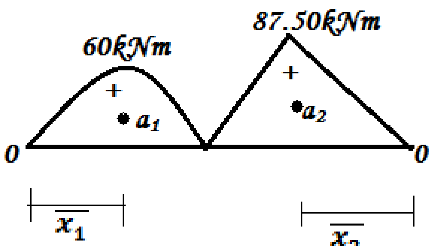
I. **Free BMD for each span**

For span AB

$$M_{AB} = \frac{wl^2}{8} = \frac{30\times4^2}{8} = 60kNm$$

Span BC

$$M_{BC} = \frac{Wab}{l} = \frac{60\times2.5\times3.5}{6} = 87.50kNm$$



Free BMD

II. **Support moments:**

Since end A is fixed,

Assume an imaginary beam A^1A

of span l_0 = 0m

∴ $M_A^1 = 0$

Applying Clapeyron's three moment

Equation for span A^1A and AB

$A^1A = l_0 = 0$, $AB = l_1 = 4m$,

$$M_A\, l_0 + 2M_B\,(l_0+l_1) + M_B\, l_1 = -\left[\frac{6a_0\overline{x_0}}{l_0} + \frac{6a_1\overline{x_1}}{l_1}\right] \quad \rightarrow \text{(I)}$$

For standard cases:

$$\left[\frac{6a_0\overline{x_0}}{l_0}\right] = 0 \text{ (No load)}$$

i) Simply supported with UDL, $\frac{6a\overline{x}}{l} = \frac{wl^3}{4}$

$$\therefore \left[\frac{6a_1\overline{x_1}}{l_1}\right] = \frac{30\times 4^3}{4} = 480$$

Substituting in equation (1)

$M_A^1\times l_0 + 2M_A\,(0+4) + M_B\times 4 = -\,[0+480]$

$0 + 8M_A + 4M_B = -\,480$

$2\,M_A + M_B = -\,120 \quad \rightarrow \quad (1)$

From span AB & BC

Applying Clapeyron's three moment equation for span AB & BC

$AB = l_1 = 4m$, $BC = l_2 = 4m$

Since end C is simply supported, $M_C = 0$

$$M_A\, l_1 + 2M_A\,(l_1+l_2) + M_C\, l_2 = -\left[\frac{a_1\overline{x_1}}{l_1} + \frac{a_2\overline{x_1}}{l_2}\right] \quad \rightarrow \text{(II)}$$

For standard cases:

1. Simply supported with UDL, $\frac{6a\overline{x}}{l} = \frac{wl^3}{4}$

   $$\therefore \left[\frac{6a_1\overline{x_1}}{l_1}\right] = \frac{30\times 4^3}{4} = 480$$

2. Simply supported with non-central load, $\frac{6a\overline{x}}{l} = \frac{Wb(l^2-b^2)}{l}$

   $$\frac{6a_2\overline{x_2}}{l} = \frac{60\times 3.5\times(6^2-3.5^2)}{6} = 831.25$$

Substituting in equation (II)

$M_A\times 4 + 2M_B\,(4+6) + M_C\times 6 = -\,[480 + 831.25]$

$4M_A + 20M_B + 0 = -\,1311.25 \quad \rightarrow (2)$

Solving equation (1) & (2)

(2) $\rightarrow$ $4\,M_A + 20\,M_B = -\,1311.25$

(1) × (2) $\rightarrow$ $-\,4\,M_A - 2\,M_B = -\,240.00$

$$0 \;\; + 18\, M_B = -1071.25$$

$$M_B = \frac{-1071.25}{18} = -\,59.91\text{kNm}$$

Substituting in equation (1)

$2\, M_A + (-59.51) = -120$

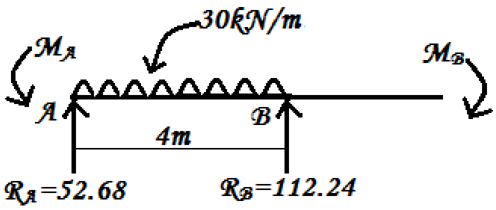
$$M_A = -\,30.25\text{kNm}$$

Final support moments:

$M_A = -\,30.25$kNm

$M_B = -\,59.51$kNm

$M_C = 0$



Reactions:

Consider span AB

Taking moment about B

$M_A + (30\times4) \times \frac{4}{2} = R_A \times 4 + M_B$

$30.25 + 240 = 4\, R_A + 59.51$

$R_A = 52.68$kN

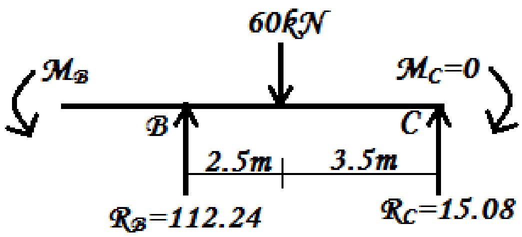
Consider span BC:

Taking moment about B:

$R_C \times 6 + M_B = M_C + 60 \times 2.5$

$6\, R_C \times 59.51 = 0 + 150$

$R_C = \frac{(150-59.51)}{6} = 15.08$kN



$\sum_V = 0$

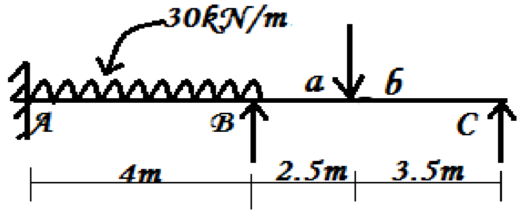
$R_A + R_B + R_C =$ Total load

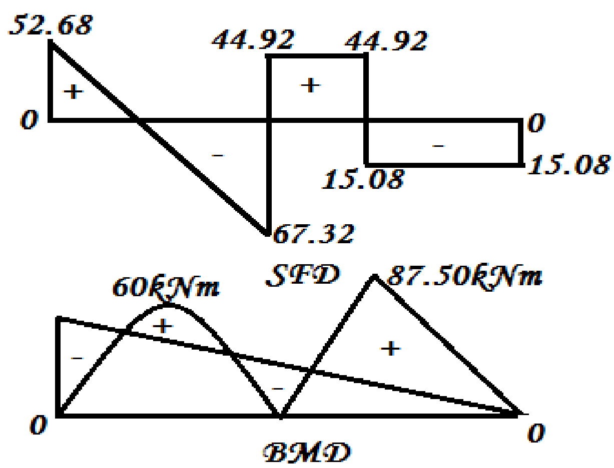
$R_B =$ Total load – $(R_A + R_C)$

$= (30\times4) + 60 – (52.68 + 15.08)$

$R_B = 112.24$kN.

Draw SFD & BMD:

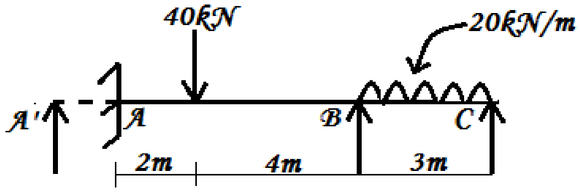


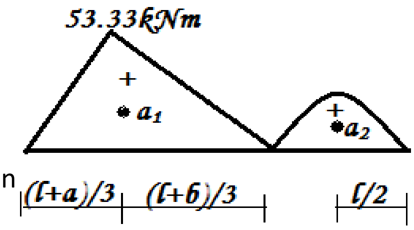


Problem 7:

A two span beam ABC of length 9m is fixed at 'A' and simply supported at 'C'. The span AB is 6m long carries a point load 40kN at 2m from A. The span BC is 3m long carries an UDL of 20kN/m. Find the support moments using Theorem of three moments method and draw SFD and BMD.

Solution:





Free BMD

i. **Free BMD:**

Span AB, a = 2m, b = 4m

$M_{AB} = \frac{Wab}{l} = \frac{40\times2\times4}{6} = 53.33\text{kNm}$

Span BC

$M_{BC} = \frac{wl^2}{8} = \frac{20\times3^2}{8} = 22.50\text{kNm}$

ii. **Support moments:**

Since end A is fixed, assume an imaginary beam A^1A of span $l_0 = 0$m.

Applying Clapeyron's three moment equation for span A^1A and AB

$A^1A = l_0 = 0$, $AB = l_1 = 4$m,

$M_A\, l_0 + 2M_A\,(l_0 + l_1) + M_B\, l_1 = -\left[\frac{6a_0\bar{x}_0}{l_0} + \frac{6a_1\bar{x}_1}{l_1}\right] \rightarrow \text{(I)}$

For standard cases:

$\left[\frac{6a_0\bar{x}_0}{l_0}\right] = 0$ (no load)

Simply supported with non-central load, $\frac{6a\bar{x}}{l} = \frac{Wb(l^2-b^2)}{l}$ from end B

$\left[\frac{6a_1\bar{x}_1}{l_1}\right] = \frac{40\times4(6^2-4^2)}{6} = 533.33$

Substituting in equation (I)

$M_A{}^1 \times l_0 + 2M_A\,(0+6) + M_B \times 6 = -[0 + 533.33]$

$0 + 12\,M_A + 6\,M_B = -533.33$

$2\,M_A + M_B = -88.89 \quad \rightarrow (1)$

Applying Clapeyron's three moment

Equation for span AB and BC

$AB = l_1 = 6$m, $BC = l_2 = 3$m,

Since end C is simply supported, $M_C = 0$

$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right] \rightarrow \text{(II)}$

For standard cases

Simply supported with non-central load, $\frac{6a\bar{x}}{l} = \frac{Wa(l^2-a^2)}{l}$ from end A.

$$\left[\frac{6a_1\bar{x}_1}{l_1}\right] = \frac{40\times2(6^2-2^2)}{6} = 426.67$$

i. Simply supported with UDL, $\frac{6a\bar{x}}{l} = \frac{wl^3}{4}$

$$\therefore \left[\frac{6a_2\bar{x}_2}{l_2}\right] = \frac{20\times3^3}{4} = 135$$

Substituting in equation (II)

$M_A \times 6 + 2\, M_B\, (6+3) + 0 = -\,[426.67 + 135]$

$6\, M_A + 18\, M_B = -\,561.67$

$M_A + 3\, M_B = -\,93.61 \rightarrow (2)$

Solving equation (1) and (2)

(1) $\rightarrow 2\, M_A + M_B = -\; 88.890$

(2) $\times 2 \rightarrow 2\, M_A + 6\, M_B = -\,187.223$

0 $-\,5\, M_B = -\; 98.33$

$M_B = -\,19.67$kNm

Substituting in equation (1)

$2\, M_A + (-19.67) = -\,88.89$

$M_A = -\,34.61$kNm

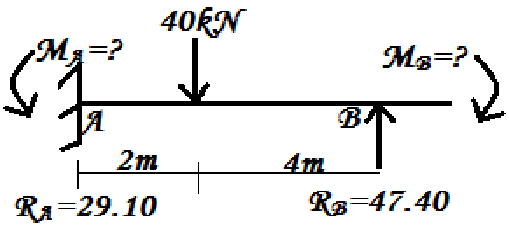
$M_A = -\,34.61$kNm

$M_B = -\,19.67$kNm

$M_C = 0$

iii. <u>Reactions:</u>

Consider span AB taking moment about B

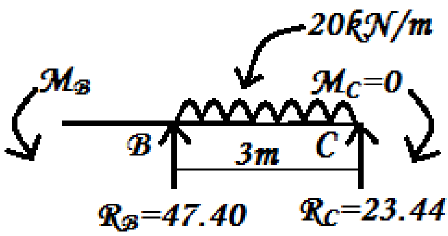


$M_A + W.\, b = M_B + R_A \times l$

$34.61 + (40\times4) = 19.67 + R_A \times 6$

$R_A = 29.16$kN

Consider span BC taking moment about B



$R_C \times l + M_B = M_C + \frac{wl^2}{2}$

$R_C \times 3 + 19.67 = 0 + \frac{20\times3^5}{2}$

$R_C = \frac{(90-19.67)}{3} = 23.44$kN

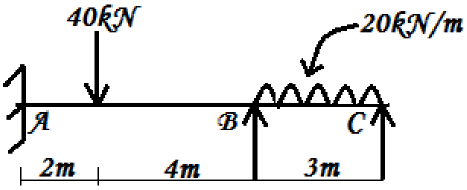
$\sum V = 0$

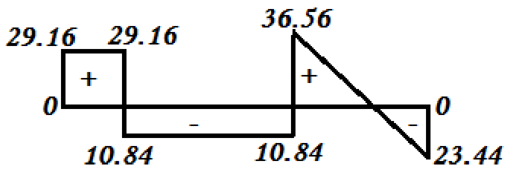
R_B = Total load – ($R_A + R_C$)

$R_B = 40 + (20\times3) - (29.16 + 23.44)$

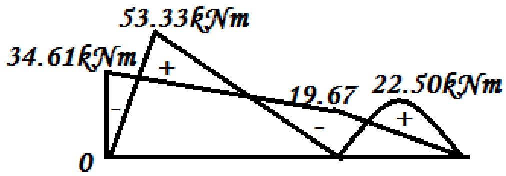
$R_B = 47.40kN$

Draw SFD and BMD





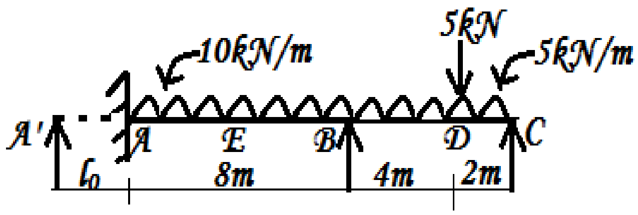
SFD

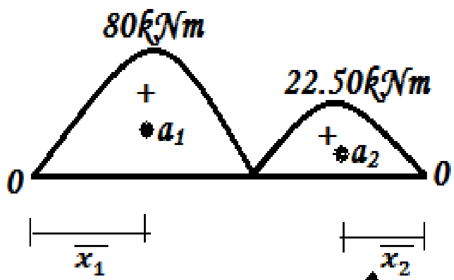


BMD

Problem 8:

A continuous beam ABC of uniform section, with span AB as 6m is fixed at A and simply supported at B and C. The beam carries an UDL of 10kN/m and span BC carries an UDL of 5kN/m. It also carries a point load of 20kN at 2m from the end C. Find the support moments using three moment equation. Draw SFD and BMD.

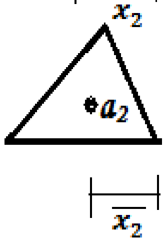




Solution:

i. **Free BMD:**

Span AB



$M_{AB} = \frac{wl^2}{8} = \frac{10\times8^2}{8} = 80kNm$

Span BC

l = 6m; a = $\frac{l}{3} = \frac{6}{3}$ = 2m

Due to UDL = $\frac{wl^2}{8} = \frac{5\times6^2}{8}$ = 22.50kNm

Due to central point load = $\frac{Wab}{l} = \frac{20\times4\times2}{6}$ = 26.67kNm

Draw free BMD

ii. **Support moments:**

End A is fixed, hence assume an imaginary beam A^1A of span l_0 = 0m

Hence assume an imaginary beam of A^1A & C^1C

Applying theorem of three moments equation span A^1A & AB,

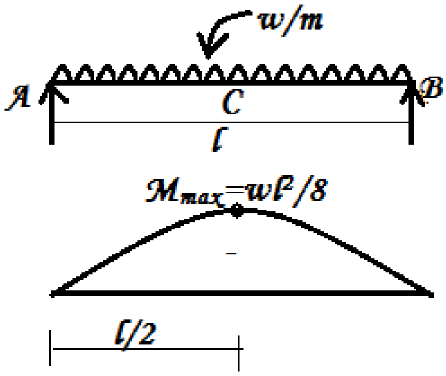
$A^1A = l_0 = 0m \quad M_{A'} = 0$

$AB = l_1 = 8m$

$$M_{A'}\, l_0 + 2M_A\,(l_0 + l_1) + M_B\, l_1 = -\left[\frac{6a_0\bar{x}_0}{l_0} + \frac{6a_1\bar{x}_1}{l_1}\right] \rightarrow (I)$$

$$\left[\frac{6a_0\bar{x}_0}{l_0}\right] = 0;$$

For standard cases:



Simply supported beam with UDL,

$$\left[\frac{6a_1\bar{x}_1}{l_1}\right] = \frac{wl^3}{4} = \frac{10\times8^3}{4} = 1280$$

Substituting in equation (I)

$0 + 2\,M_A\,(0+8) + M_B \times 8 = -\,[0+1280]$

$16\,M_A + 8\,M_B = -\,1280$

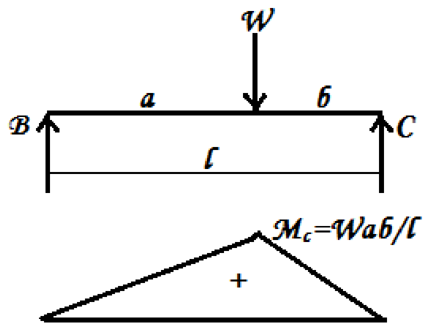
$2\,M_A + M_B = -\,160 \quad \rightarrow (1)$

Applying theorem of three moments equation for span AB and BC

$AB = l_1 = 8m,\ BC = l_2 = 6m,$

Since 'C' is simply supported $M_C = 0$

$$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right] \rightarrow (II)$$



For standard cases:

Simply supported beam with UDL

$$\frac{6a\bar{x}}{l} = \frac{wl^3}{4} = 384$$

$$\frac{6a_1\overline{x_1}}{l_1} = \frac{10\times8^3}{4} = 1280$$

Simply supported beam with non-central load

l = 6m, a = 4m, b = 2m,

$$\frac{6a\overline{x}}{l} = \frac{Wb(l^2-b^2)}{l} \text{ from end 'C'}$$

$$\frac{6a_2\overline{x_2}}{l_2} = \left[\frac{Wl^3}{4} + \frac{Wb(l^2-b^2)}{l}\right] = \left[\frac{5\times6^3}{4} + \frac{20\times2(6^2-2^2)}{6}\right]$$

= [270 + 2013.33] = 483.33

Substituting in equation (II)

$M_A \times 8 + 2\, M_B = -\,[1280 + 483.33] = -\,1763.33$

$8\, M_A + 28\, M_B = -\,1763.33 \quad \rightarrow (2)$

Solving equations (1) and (2)

1) $\times\, 28 \rightarrow 56\, M_A + 28\, M_B = -\,4480.00$
2) $\rightarrow\; 8\, M_A + 28\, M_B = -\,1763.33$

$48\, M_A + 0 \quad = -\,2716.70$

$M_A = -\,56.60$kNm

Substituting in (1)

$2\, M_A + M_B = -\,160 \quad \rightarrow (1)$

$2 \times (-\,56.60) + M_B = -\,160$

$M_B = -\,160 + 113.2 = -\,46.80$

$M_B = -\,46.80$kNm

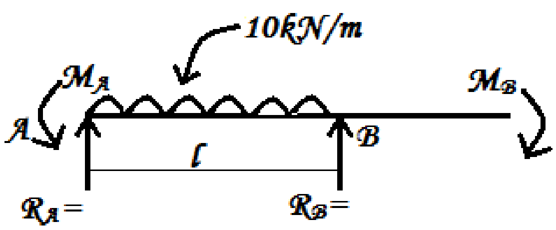
III. <u>Result:</u>

Final Support Moments:

$M_A = -\,56.60$kNm

$M_B = -\,46.80$kNm

$M_C = 0$

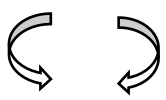


Draw BMD as shown in fig.

Reactions:

Consider span AB

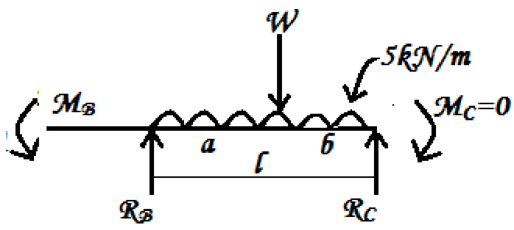
Taking moment at B



$M_A + \frac{wl^2}{2} = R_A \times l + M_B$

$56.60 + \frac{10\times8^2}{2} = 8\,R_A + 46.80$

$R_A = 41.23kN$



Consider span BC:

Taking moment about B

$R_C \times l + M_B = M_C + (W \times a) + \frac{wl^2}{2}$

$R_C \times 6 + 46.80 = 0 + (20 \times 4) + \frac{5\times6^2}{2}$

$R_C = 20.53kN$

$\sum V = 0$

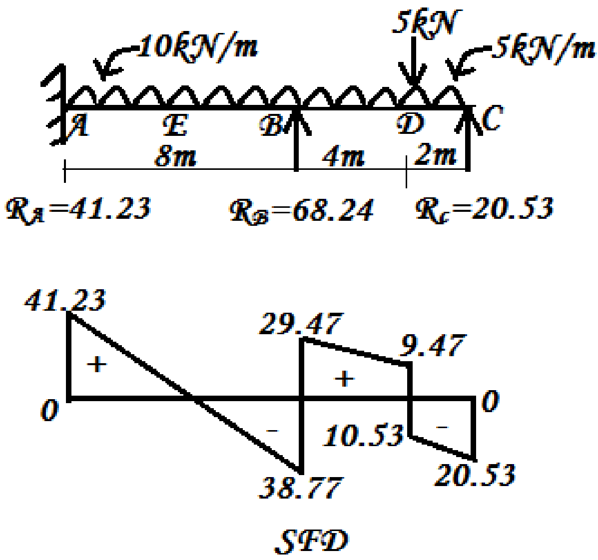
$R_A + R_B + R_C =$ Total load

$R_B =$ Total load $- (R_A + R_C)$

$41.23 + R_B + 20.53 = (10\times8) + (5\times6) + 20$

$R_B = 68.24kN.$

Draw SFD



Net BMD:

$M_C = 0$

$M_D = 20.53 \times 2 \left(\frac{5\times2^2}{2}\right) = 31.06kNm$

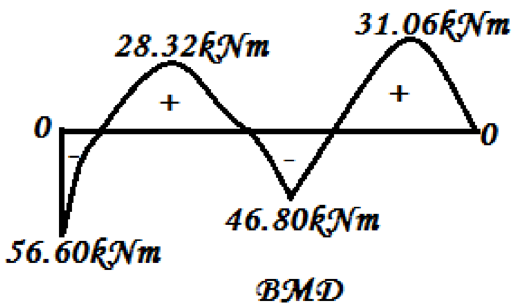
$M_B = -\,46.80kNm$

$M_E = R_A \times 4 - M_A - \frac{wl^2}{2}$

$= R_A \times 4 - 56.60 - \frac{10\times4^2}{2}$

$= (164.92 - 56.60 - 80) = 28.32kNm$

$M_A = -\,56.60kNm$

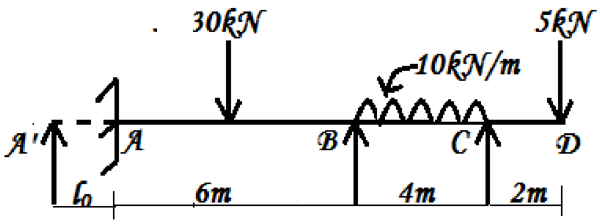


Type – IV One End Fixed and Other End With Over Hanging

Problem 9:

A continuous beam ABCD is fixed at A and simply supported at B and C. Span AB is 6m long and carries point load of 30kN at its mid span, the span BC is 4m long carries an UDL of 10kN/m over its entire span, span CD is 2m long carries a point load of 5kN at free end D. Find the support moments by Clapeyron's theorem of three moments. Draw SFD and BMD.

Solution:



I. **Free BMD:**

Span AB

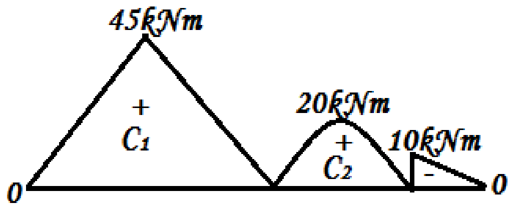
$$M_{AB} = \frac{Wl}{4} = \frac{10\times6}{4} = 45\text{kNm}$$

Span BC

$$M_{BC} = \frac{wl^2}{8} = \frac{10\times4^2}{8} = 20\text{kNm}$$

Support 'C' is with overhanging

$\therefore$ $M_{CD} = -5 \times 2 = -10\text{kNm}$



Free BMD

II. **Support moments:**

End A is fixed, introduce an imaginary beam A^1A of span

$l_0 = 0\text{m}$; $M_A^{\,1} = 0$; $a_0 = 0$

Applying theorem of three moments equation span A^1A and AB

$A^1A = l_0$, $AB = l_1 = 6\text{m}$

$$M_A\, l_0 + 2M_A\,(l_0 + l_1) + M_B\, l_1 = -\left[\frac{6a_0\bar{x}_0}{l_0} + \frac{6a_1\bar{x}_1}{l_1}\right] \rightarrow \text{(I)}$$

$$\left[\frac{6a_0\bar{x}_0}{l_0}\right] = 0$$

For standard cases:

Simply supported with central load, $\left[\frac{6a\bar{x}}{l}\right] = \frac{3}{8}\,Wl^{\,2}$

$$\left[\frac{6a_1\bar{x}_1}{l_1}\right] = \frac{3}{8} \times 30 \times 6^2 = 405$$

Substituting in equation (I)

$$0 + 2\,M_A\,(0+6) + M_B \times 6 = -\,[0+405]$$

$$12\,M_A + 6\,M_B = -\,405 \quad \rightarrow (1)$$

(1) Applying theorem of three moments equation for span AB and BC.

$AB = l_1 = 6\text{m}$, $BC = l_2 = 4\text{m}$,

$M_C = - 10kNm$

$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = - \left[\frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}\right] \rightarrow$ (II)

For standard cases

Simply supported with central load, $\left[\frac{6a\overline{x}}{l}\right] = \frac{3}{8} Wl^2$

$\left[\frac{6a_1\overline{x_1}}{l_1}\right] = \frac{3}{8} \times 30 \times 6^2 = 405$

Simply supported with UDL, $\left[\frac{6a\overline{x}}{l}\right] = \frac{wl^3}{4}$

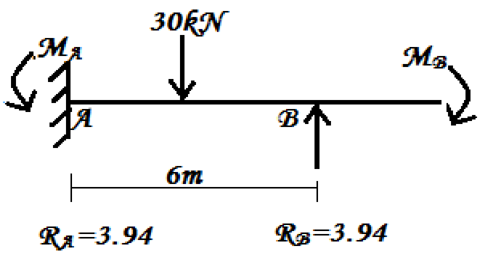
$\frac{6a_2\overline{x_2}}{l} = \frac{wl^3}{4} = \frac{10\times4^3}{4} = 160$

Substituting in equation (II)

$6\, M_A + 2\, M_B\,(6+4) + 4 \times 10 = - [405 + 160]$

$6\, M_A + 20\, M_B = - 65 + 40 = - 525$

$6\, M_A + 20\, M_B = - 525 \rightarrow (2)$



Solving equation (1) and (2)

1. $12\, M_A + 6\, M_B = - 405$
2. $\times 3 \rightarrow 12\, M_A + 40\, M_B = - 1050$

$34\, M_B = - 645$

$M_B = - 18.97kNm$

Substituting in equation (2)

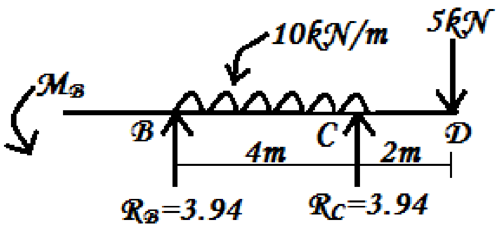
$6\, M_A + 20 \times (- 18.97) = - 525$

$6\, M_A - 379.40 = - 525$

$M_A = \frac{-525 + 379.40}{6} = - 27.67kNm$

$M_A = - 24.27kNm$

III. Reactions:



Consider span AB

Taking moment about left of 'B'

$M_A + W\frac{l}{2} = R_A \times l + M_B$

$24.27 + 30 \times \frac{6}{2} = 6\, R_A + 18.97$

$$6\,R_A = 99.85$$

$$R_A = 16.47kN$$

Consider span BCD

Taking moment about right of B

$$R_C \times l + M_B = w\,l\,\frac{l}{2} + W\,(l+a)$$

$$4R_C + 18.97 = 10 \times 4 \times \frac{5}{2} + 5\,(4+2)$$

$$4R_C + 18.97 = 110$$

$$R_C = \frac{(92-8)}{4} = 22.75kN$$

$$\sum V = 0$$

$$R_B = \text{Total load} - (R_A + R_C)$$

$$R_B = 30 + (10\times4) + 5 - (16.47 + 22.75)$$

$$R_B = 35.78kN$$

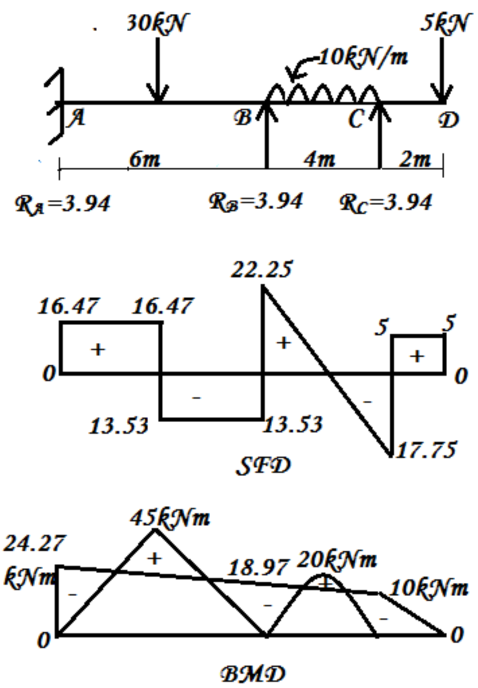
Final support moments:

$M_A = -\,24.27kNm$

$M_B = -\,18.97kNm$

$M_C = -\,10kNm$

Draw SFD and BMD:



Type – V Both Ends Fixed

Problem 10:

A continuous beam ABC is fixed at A and C simply supported at B each span AB and BC is 6m. The span AB carries an UDL of 20kN/m and span BC carries a point load of 60kN at mid span. Using theorem of three moments find support moments. Draw SFD and BMD.

Solution:

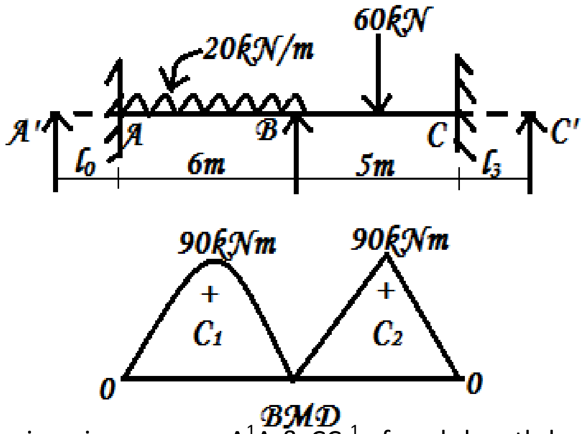
i. **Free BMD:**

Span AB

$$M_{AB} = \frac{wl^2}{8} = \frac{20\times6^2}{8} = 90\text{kNm}$$

Span BC

$$M_{BC} = \frac{Wl}{4} = \frac{60\times6}{4} = 90\text{kNm}$$



ii. **Support moments:**

Since ends A and C are fixed, assume an imaginary spans A^1A & CC^1 of each length l_0 = 0m and l_3 = 0m

$M_{A'} = 0$; $M_{C'} = 0$

Applying theorem of three moments for span A^1A and AB

$$M_A^1\, l_0 + 2M_A\,(l_0 + l_1) + M_B\, l_1 = -\left[\frac{6a_0\bar{x}_0}{l_0} + \frac{6a_1\bar{x}_1}{l_1}\right] \rightarrow \text{(I)}$$

$$\left[\frac{6a_0\bar{x}_0}{l_0}\right] = 0$$

For standard cases:

Simply supported with UDL, $\left[\frac{6a\bar{x}}{l}\right] = \frac{wl^3}{4}$

$$\left[\frac{6a_1\bar{x}_1}{l_1}\right] = \frac{wl^3}{4} = \frac{20\times6^3}{4} = 1080$$

Substituting in equation (I)

$$M_A\, l_0 + 2M_A\,(0 + 6) + M_B \times 6 = -[0 + 1080]$$

$$0 + 12\,M_A + 6\,M_B = -1080$$

$$2\,M_A + M_B = -1080 \quad \rightarrow (1)$$

(1) Applying theorem of three moments equation for span AB and BC.
AB = l_1 = 6m, BC = l_2 = 5m,

$$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right] \rightarrow \text{(II)}$$

For standard cases:

Simply supported with UDL, $\left[\frac{6a\bar{x}}{l}\right] = \frac{wl^3}{4}$

$$\left[\frac{6a_1\bar{x}_1}{l_1}\right] = \frac{wl^3}{4} = \frac{20\times6^3}{4} = 1080$$

Simply supported with central point load, $\left[\frac{6a\bar{x}}{l}\right] = \frac{3}{8}\,Wl^2$

$$\left[\frac{6a_2\bar{x}_2}{l_2}\right] = \frac{3}{8}\,Wl^2 = \frac{3}{8} \times 60 \times 6^2 = 810$$

Substituting in equation (II)

$M_A \times 6 + 2\,M_B\,(6+6) + 6\,M_C = -\,[1080 + 810]$

$6\,M_A + 24\,M_B + 6\,M_C = -\,1890$

$M_A + 4\,M_B + M_C = -\,315 \quad \rightarrow (2)$

Applying theorem of three moments for span BC and CC ^{1}

$BC = l_2 = 6m;\ CC^1 = l_3 = 0m,\ M_C^1 = 0$

$M_B\,l_2 + 2M_C\,(l_2 + l_3) + M_C^1\,l_3 = -\left[\frac{6a_2\bar{x}_2}{l_2} + \frac{6a_3\bar{x}_3}{l_3}\right] \quad \rightarrow (III)$

We know,

$$\left[\frac{6a_2\bar{x}_2}{l_2}\right] = \frac{3}{8}\,Wl^2 = \frac{3}{8} \times 60 \times 6^2 = 810\text{kNm from end C}$$

$$\left[\frac{6a_3\bar{x}_3}{l_3}\right] = 0$$

Substituting in equation III

$M_B \times 6 + 2\,M_C\,(6+0) + 0 = -\,[810 + 0]$

$6\,M_B + 12\,M_C = -\,810$

$M_B + 2\,M_C = -\,135 \quad \rightarrow (3)$

Solving equations (2) and (3)

(3) $\rightarrow$ $M_B + 2\,M_C = -\,135$

(2) $\times 2 \rightarrow 2\,M_A + 8\,M_B + 2\,M_C = -\,630$

(Solving 2&3) $-\,2\,M_A - 7\,M_B + 0 = 495$

$2\,M_A + 7\,M_B + 0 = -\,495 \quad \rightarrow (4)$

Solving (1) and (4)

(4) $\rightarrow$ $2\,M_A + 7\,M_B = -\,495$

(1) $\rightarrow 2\,M_A + M_B = -\,180$

$6\ M_B = -\ 315$

$$M_B = \frac{-\ 315}{6} = -\ 52.50$$

$M_B = -\ 52.50 kNm$

Substituting in (1)

$2\ M_A + (-52.50) = -\ 180$

$$M_A = \frac{(-180 + 52.5)}{2} = -\ 63.75 kNm$$

$M_A = -\ 63.75 kNm$

Substituting value of M_B in equation (3)

$M_B + 2\ M_C = -\ 135 \rightarrow (3)$

$-\ 52.50 + 2\ M_C = -\ 135$

$M_C = -\ 41.25 kNm$

Reactions:

<table>
<tr><td>support</td><td>A</td><td colspan="2">B</td><td>C</td></tr>
<tr><td>Reaction due to simply supported beam $\frac{wl}{2}, \frac{W}{2}$</td><td>+60</td><td>+60</td><td>+30</td><td>+30</td></tr>
<tr><td>Reactions due to fixing moments $\frac{M_A \sim M_B}{l_1}, \frac{M_B \sim M_C}{l_2}$</td><td>-1.88</td><td>+1.87</td><td>+1.88</td><td>-1.87</td></tr>
<tr><td rowspan="2">Final reactions
R_A, R_B & R_C =</td><td rowspan="2">+ 58.12
R_A</td><td>61.87</td><td>31.88</td><td rowspan="2">28.13
R_C</td></tr>
<tr><td colspan="2">93.75 R_B</td></tr>
</table>

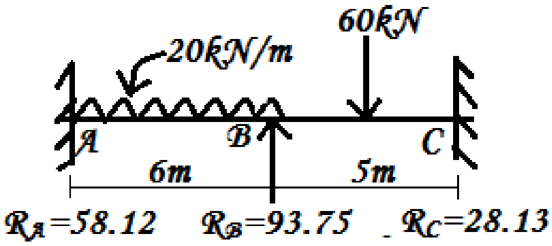
Result:

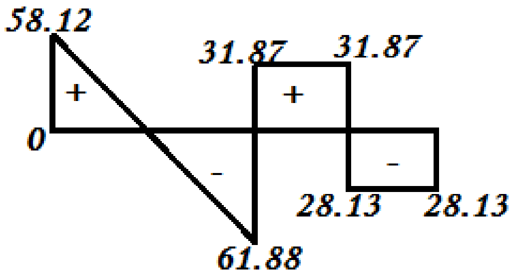
$M_A = -\ 63.75 kNm$

$M_B = -\ 52.50 kNm$

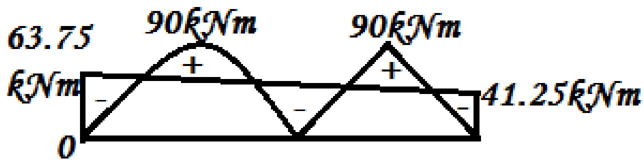
$M_C = -\ 41.25 kNm$

Draw SFD and BMD





SFD

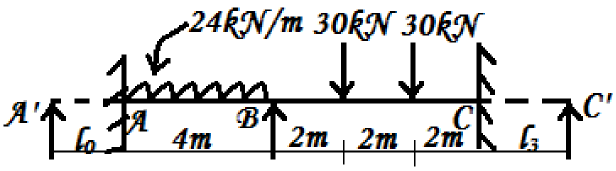


BMD

Problem 11:

A two span continuous beam of 4m and 6m spans are fixed at both of its end. The size of the beam is uniform in both spans. The span AB is 4m carries an UDL of 24kN/m throughout its length. The 6m span carries two point loads of 30kN each at its one third points. Find out the support moments by Clapeyron's theorem of three moments.

Solution:



I. **Free BMD**

Span AB

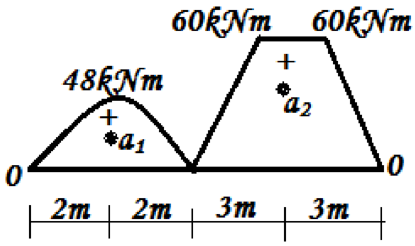
$M_{AB} = \frac{wl^2}{8} = \frac{24\times4^2}{8} = 48\text{kNm}$

Span BC

$l = 6\text{m};\ a = \frac{l}{3} = \frac{6}{3} = 2\text{m}$

$M_{BC} = w.a = 30\times2 = 60\text{kNm}$

Draw free BMD.



Free BMD

II. **Support moments:**

End A and C are fixed

Hence assume an imaginary beams of A^1A & C^1C

$A^1A = l_0 = 0\text{m};\ C^1C = l_3 = 0\text{m};\ M_{A'} = M_{C'} = 0$

Applying theorem of three moments equation Span A^1A and AB,

$AB = l_1 = 6m$

$M_{A'}\, l_0 + 2M_B\,(l_0 + l_1) + M_B\, l_1 = -\left[\frac{6a_0\overline{x_0}}{l_0} + \frac{6a_1\overline{x_1}}{l_1}\right]$

$\left[\frac{6a_0\overline{x_0}}{l_0}\right] = 0; \quad \left[\frac{6a_1\overline{x_1}}{l_1}\right] = \frac{wl^3}{4} = \frac{24\times4^3}{4} = 384$

$0 + 2\,M_A\,(0+4) + M_B \times 4 = -\,[0 + 384]$

$8\,M_A + 4\,M_B = -\,384$

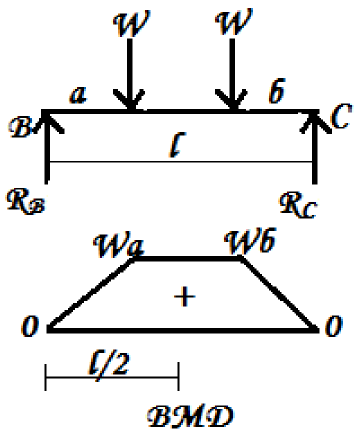
$2\,M_A + M_B = -\,96 \quad \rightarrow (1)$

Applying theorem of three moments equation for span AB and BC

$AB = l_1 = 4m,\ BC = l_2 = 6m,$

$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = -\left[\frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}\right]$

$\frac{6a_1\overline{x_1}}{l_1} = \frac{wl^3}{4} = 384$



$a_2 = \left(\frac{(a+b)}{2}\right) h = \frac{2+6}{2} \times 60 = 240$

$\overline{x_2} = \frac{6}{2} = 3m$

$\frac{6a_2\overline{x_2}}{l_2} = \frac{6\times240\times3}{6} = 720$

$4\,M_A + 2\,M_B\,(4+6) + 6\,M_C = -\,[384 + 720]$

$4\,M_A + 20\,M_B + 6\,M_C = -\,1104$

$2\,M_A + 10\,M_B + 3\,M_C = -\,552 \quad \rightarrow (2)$

Applying theorem of moments equation for span BC and CC^1

$BC = l_2 = 6m,\ CC^1 = l_3 = 0m,$

$M_B\, l_2 + 2M_C\,(l_2 + l_3) + M_C\, l_3 = -\left[\frac{6a_2\overline{x_2}}{l_2} + \frac{6a_3\overline{x_3}}{l_3}\right]$

$\frac{6a_2\overline{x_2}}{l_2} = \frac{6\times240\times3}{6} = 720; \quad \frac{6a_3\overline{x_3}}{l_3} = 0;$

$6\,M_B + 2\,M_C\,(6+0) = -\,[720+0]$

$6\,M_B + 12\,M_C = -\,720$

$M_B + 2\,M_C = -\,120 \quad \rightarrow (3)$

$2\,M_A + M_B = -\,96 \quad \rightarrow (1)$

$2\,M_A + 10\,M_B + 3\,M_C = -\,552 \quad \rightarrow (2)$

$M_B + 2\,M_C \quad = -120 \quad \rightarrow (3)$

Solving equations (1) and (2)

$(2) \rightarrow \quad 2\,M_A + 10\,M_B + 3\,M_C = -552$

$(1) \rightarrow \quad 2\,M_A + \quad 0\,M_B \quad = -96$

$0 + 9\,M_B + M_C = -456$

$3\,M_B + M_C = -152 \quad \rightarrow (4)$

Solving equation (3) and (4)

$(4) \times 2 \rightarrow 6\,M_B + 2\,M_C = -304$

$(3) \quad \rightarrow \quad M_B + 2\,M_C = -120$

$5M_B + 0 \quad = -184$

$M_B = \frac{-184}{5} = -36.80\text{kNm}$

Substituting in (3)

$M_B + 2\,M_C = -120$

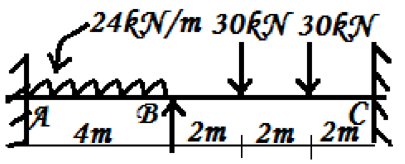
$-36.80 + 2\,M_C = -120$

$2\,M_C = -120 + 36.80$

$2\,M_C = -83.20$

$M_C = -\frac{83.20}{2} = -41.60\text{kNm}$

Substituting M_B in (1)



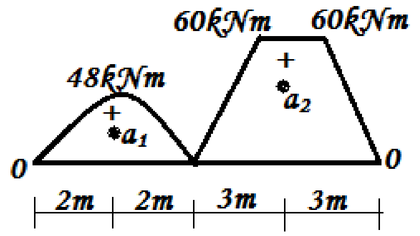
$2\,M_A + M_B = -96$

$2\,M_A - 36.80 = -96$

$2\,M_A = -96 + 36.80$

$2\,M_A = -59.20$

$M_A = \frac{-59.20}{2} = -29.60\text{kNm}$



Free BMD

Draw BMD as shown in fig.

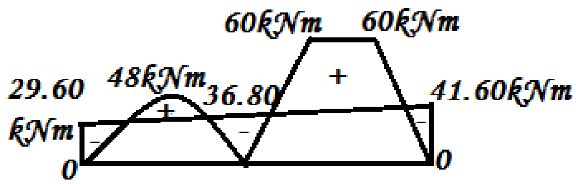
Result:

Final support moments

$M_A = -29.60\text{kNm}$

$M_B = -36.80\text{kNm}$

$M_C = -41.60\text{kNm}$



Net BMD

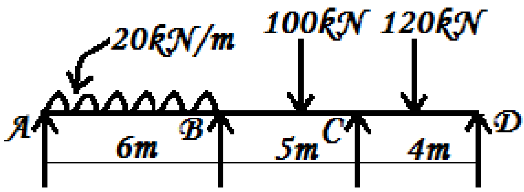
Three Span Continuous Beam

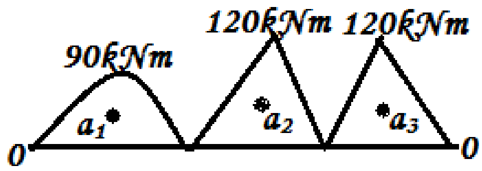
Problem 12:

Draw BMD for a continuous beam ABCD as shown in fig by theorem of three moments method.

Solution:

I. **Free BMD**





Free BMD

Span AB

$$M_{AB} = \frac{wl^2}{8} = \frac{20\times6^2}{8} = 90\text{kNm}$$

Span BC

$$M_{BC} = \frac{Wab}{l} = \frac{100\times3\times2}{5} = 120\text{kNm}$$

Span CD

$$M_{CD} = \frac{Wl}{4} = \frac{120\times4}{4} = 120\text{kNm}$$

II. **Support moments:**

Since ends A and D are simply supported

$M_A = 0;\ M_D = 0$

Applying theorem of three moments for span AB and BC

$AB = l_1 = 6\text{m},\ BC = l_2 = 5\text{m},$

$$M_A\, l_1 + 2M_B\,(l_1 + l_2) + M_C\, l_2 = -\left[\frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}\right] \rightarrow \text{(I)}$$

For standard cases

Simply supported with UDL, $\left[\frac{6a\bar{x}}{l}\right] = \frac{wl^3}{4}$ from end 'A'

$$\frac{6a_1\bar{x}_1}{l_1} = \frac{wl^3}{4} = \frac{20\times6^3}{4} = 1080$$

Simply supported beam with non-central load

$l = 5\text{m},\ a = 3\text{m},\ b = 2\text{m},$

$\left[\frac{6a\bar{x}}{l}\right] = \left[\frac{Wb(l^2-b^2)}{l}\right]$ from end 'C'

$$\frac{6a_2\bar{x}_2}{l_2} = \left[\frac{Wb(l^2-b^2)}{l}\right] = \left[\frac{100\times2(5^2-2^2)}{5}\right] = 840$$

Applying theorem of three moments:

$$0 + 2\,M_B\,(6+5) + M_C\times5 = -\,[1080 + 840]$$

$$22\,M_B + 5\,M_C = -\,1920 \qquad \rightarrow (1)$$

Applying theorem of three moments for span BC and CD

BC = l_2 = 5m, CD = l_3 = 4m,

$M_D = 0$

$M_B l_2 + 2M_C (l_2 + l_3) + M_D l_3 = -\left[\frac{6a_2\overline{x_2}}{l_2} + \frac{6a_3\overline{x_3}}{l_3}\right]$ → (II)

For standard cases:

Simply supported beam with non-central load

l = 5m, a = 3m, b = 2m,

$\left[\frac{6a\overline{x}}{l}\right] = \left[\frac{Wa(l^2-a^2)}{l}\right]$ from end 'B'.

$\frac{6a_2\overline{x_2}}{l_2} = \left[\frac{Wa(l^2-a^2)}{l}\right] = \left[\frac{100\times3(5^2-3^2)}{5}\right] = 960$

Simply supported with central point load, $\left[\frac{6a\overline{x}}{l}\right] = \frac{3}{8}\ Wl^2$ from end D

$\left[\frac{6a_2\overline{x_2}}{l_2}\right] = \frac{3}{8}\ Wl^2 = \frac{3}{8}\times120\times4^2 = 720$

Substituting in equation II

$5\ M_B + 2\ M_C (5+4) + 0 = -[960 + 720]$

$5\ M_B + 18\ M_C = -1680$ → (2)

Solving equations (1) and (2)

(2) × 22 → $110\ M_B + 396\ M_C = -36960$

(1) × 2 → $-110\ M_B + 25\ M_C = -9600$

$0 + 371\ M_C = -27360$

$M_C = \frac{-27360}{371} = -73.75$

M_C = - 73.75kNm

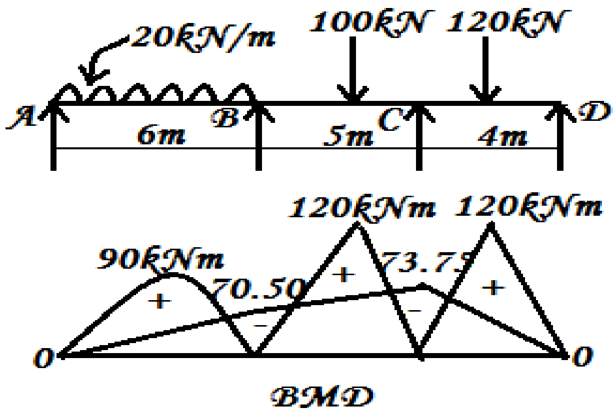
Substituting value of M_C in equation (3)

$5\ M_B + 18\times(-73.75) = -1680$

$5\ M_B = -1680 + 1327.5 = -352.50$

$M_B = \frac{-352.50}{5} = -70.50$

M_B = - 70.50kNm



Draw SFD and BMD:

Result:

Final support moments:

$M_A = 0$

$M_B = + 70.50kNm$

$M_C = - 73.75kNm$

$M_D = 0$

Propped Cantilever Beam

(By theorem of three moments method)

Problem 13:

A cantilever beam of span 6m is propped at 2m from free end it carries an UDL of 12kN/m over its entire span. Determine

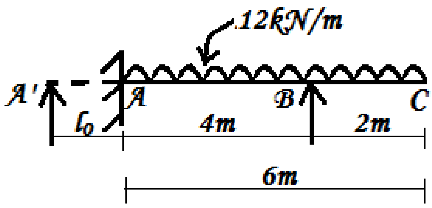
a. The prop. Reaction

b. Support moments and

c. Draw SFD and BMD by theorem of three moment method.

Given:

Span l = 4m; Projection = 2m; UDL, w =12kN/m

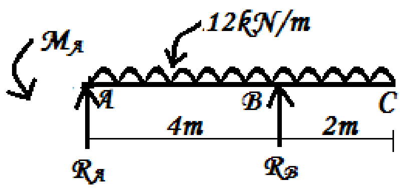


Solution:

i. **Free BMD:**

Span AB

$M_{max} = \frac{wl^2}{8} = \frac{12\times4^2}{8} = 24kNm;$



Span BC

$M_B = -\frac{wX^2}{2} = \frac{12\times2^2}{2} = -24kNm.$

ii. **Support moments:**

Since end A is fixed assume an imaginary beam of A^1A

$A^1A = l_0 = 0m;$

Applying theorem of three moments equation of span A^1A

$M_A^{\ 1} = 0m; AB = l_1 = 4m$

$M_{A'}\, l_0 + 2M_B\,(l_0 + l_1) + M_B\, l_1 = -\left[\frac{6a_0\bar{x}_0}{l_0} + \frac{6a_1\bar{x}_1}{l_1}\right] \rightarrow (I)$

For standard cases:

$\left[\frac{6a_0\bar{x}_0}{l_0}\right] = 0;\ \left[\frac{6a_1\bar{x}_1}{l_1}\right] = \frac{wl^3}{4} = \frac{12\times4^3}{4} = 192$

$0 + 2\, M_A\,(0+4) + M_B \times 4 = -\,[0 + 192]$

$M_A + 4\ M_B = -\ 192$

$2\ M_A + 4(-24) = -\ 192$

$M_A = \frac{-192+96}{2} = -\ 48kNm$

iii. <u>Prop reaction:</u>

Let R_B = prop reaction at B; R_A = reaction at A

Taking moment about A

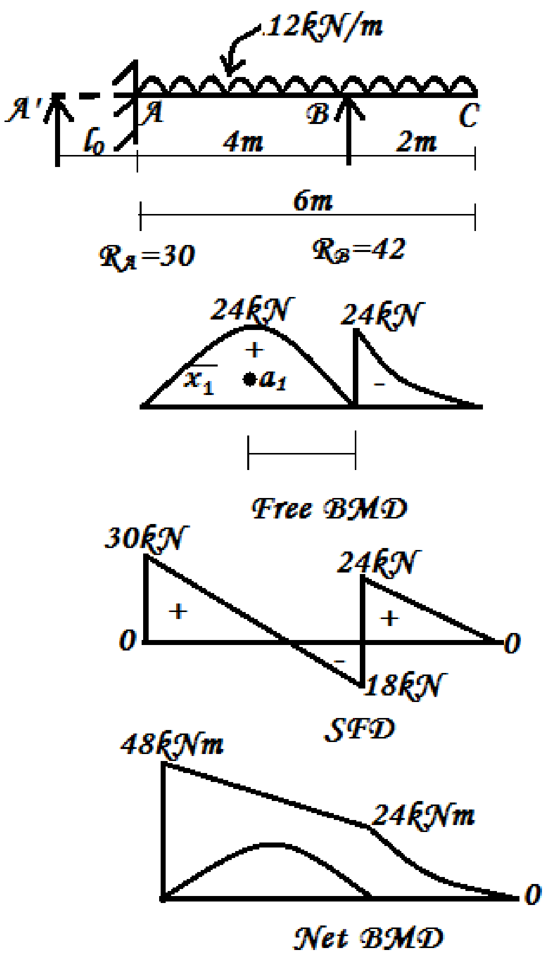
$R_B \times 4 + M_A = 12 \times 6 \times \frac{6}{2}$

$4\ R_B + 48 = 216$

$R_B = \frac{-192+96}{2} = 42kNm$

R_A = Total load – R_B = (12×6 – 42) = 30kNm

Draw SFD and BMD as shown in fig



iv. <u>Results:</u>

$M_A = -\ 48kNm$; $M_B = -\ 24kNm$

$R_A = 30kNm$; $R_B = 42kNm$.

HIGH LIGHTS

Statement of Clapeyron's Theorem of three moments.

Clapeyron's theorem states that if a beam has 'n' supports, the end being fixed then the same number of equations required to determining the support moments may be obtained from the consecutive pairs of spans i.e. AB-BC, BC-CD, CD-DE and so on. Continuous beam ABC loaded as shown in fig.

$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = -\left(\frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}\right)$

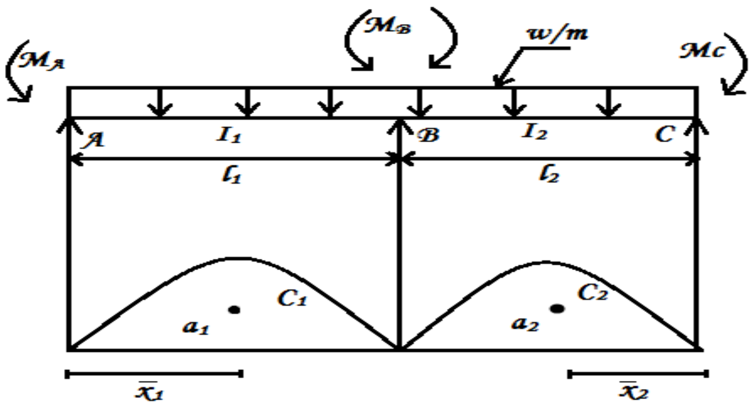
Where,

a_1 = area of free BMD for span AB

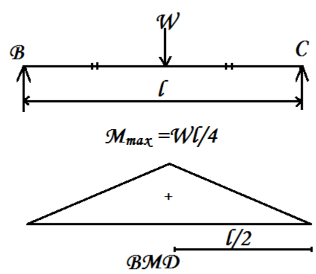
a_2 = area of fixed BMD for span BC

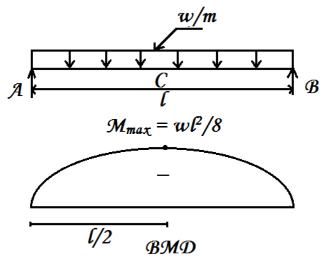
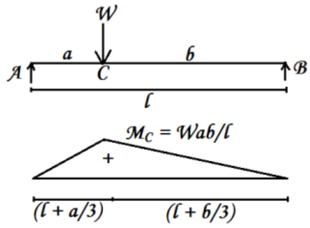
$\overline{x}_1$ = c.g. of BMD from left end (A)

$\overline{x}_2$ = c.g. of BMD from right end (C)



Area of Free BMD for standard cases

Sl.no	Types of Beam	$\frac{6a\bar{x}}{l}$ from left end	$\frac{6a\bar{x}}{l}$ from right end	BMD
1.	Simply supported beam with central point load	$\frac{3}{8}Wl^2$	$\frac{3}{8}Wl^2$	

2.	Simply supported beam with UDL of w/m	$\frac{wl^3}{4}$	$\frac{wl^3}{4}$	
3.	Simply supported beam with non-central load	$\frac{Wa(l^2-a^2)}{l}$	$\frac{Wb(l^2-b^2)}{l}$	

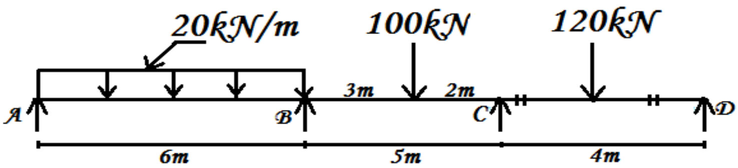
QUESTIONS

Two mark questions

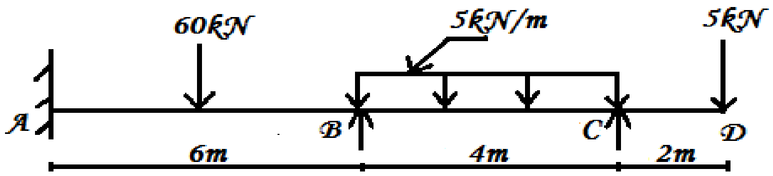
1. State the degree of indeterminacy of a fixed beam
2. When a beam is called indeterminate?
3. Give two examples of indeterminate beams.
4. Where the (-ve) moment is maximum in a two span continuous beam having simple supports at the ends?
5. State the application of theorem of three moment equation for continuous beams with fixed ends.

Ten mark questions

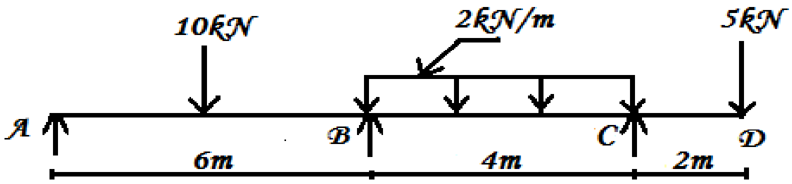
1. A continuous beam ABCD of length 12m is simply supported by three supports at A, B and C with an equal spacing of 5m. It carries an UDL of intensity 20kN/m over the two spans. There is 30kN load on the free end, D. Analyse the beam using Clapeyron's Theorem and draw the SF and BM diagram.
2. A continuous beam ABC is simply supported at A and C such that AB = 6m and BC = 5m. The span AB carries an UDL of 20kN/m and the span BC carries a point load of 50kN at its mid - span. Find the support moments by theorem of three moments. Draw the BMD and SFD.
3. A continuous beam ABC of length 8m has two equal spans. The AB carries an UDL of 20kN/m over its entire length and span BC carries a point load of 20kN at 3m from B. Draw the BMD and SFD. Apply theorem of three moments method. End A & C are simply supported.
4. A continuous beam ABCD of length 9m is fixed at A and simply supported at C. The span AB of length 6m carries a point load of magnitude 40kN at 2m from A. The span BC of length 3m carries an UDL of intensity 20kN/m. The size of beam is uniform throughout its length. Analyse the beam using three moments and draw the BMD and SFD diagrams.
5. Analyse the continuous beam shown in fig. By the use of Clapeyron's theorem of three moments. Draw the BMD.



6. Analyse the continuous beam shown in fig. by the use of Clapeyron's Theorem of three moments. Take EI constants. Draw the BMD.



7. Determine the support moments for the beam shown in figure by Clapeyron's theorem of three moments. Draw the bending moment diagram. EI is constant.



8. A continuous beam ABC of length 8m has two equal spans. The span AB carries an UDL of 20kN/m over its entire length and the span carries a point load of 20kN at 3m from B. Draw SFD and BMD. Take ends A & C are simply supported. Apply theorem of three moments.

9. A continuous beam ABC of span 10m is fixed at end A and simply supported at C span AB is 4m long and carries an UDL of 30kN/m over entire span and span BC carries a point load of 60kN at 2.5m from B. Determine the support moments by using theorem of three moments. Draw BMD.

10. A continuous Beam ABC is fixed at A and C simply supported at B each span AB and BC is 6m. The span AB carries an UDL of 20kN/m and span BC carries a point load of 60kN at mid - span. Using theorem of three moments find support moments. Draw SFD and BMD.

11. A continuous beam ABC of uniform section, with span AB as 6m is fixed at A and simply supported at B and C. The beam carries an UDL of 10kN/m and span BC carries an UDL of 5kN/m. it also carries a point load of 20kN at 2m from the end C. Find the support moments using three moment equation. Draw SFD and BMD.

3.1 CONTINUOUS BEAMS – MOMENT DISTRIBUTION METHOD

Introduction:

The moment distribution method was first introduced by Prof. Hardy cross, an American structural engineer in 1930. It is also popularly known as Hardy cross method and widely used for the analysis of all indeterminate structure like continuous beams and portal frames.

Concept:

- The moment distribution method consists of successive approximations using a series of cycles, each converging towards a precise final result.
- It is initially assumed that all the joints are fixed or clamped and then the fixed end moments (FEM) due to external loads are calculated and those calculated moments at every joint are checked for equilibrium after releasing the initially introduced clamps by applying equal and opposite moment to balance a joint and evaluating its effects on opposite joints.

The process is repeated till the required accuracy is got.

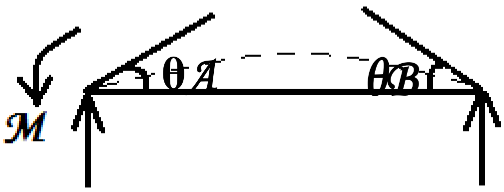
Fundamental concepts:

1) Beam stiffness
2) Relative stiffness or stiffness ratio
3) Distribution factor
4) Distribution moment
5) Carryover moment
6) Carryover factor

1). Beam stiffness:

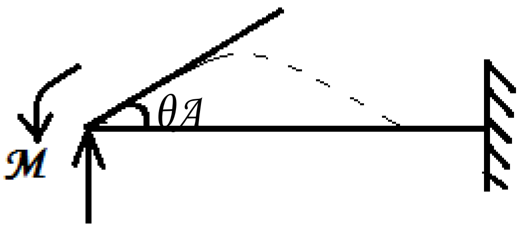
Stiffness is a measure of resistance of a structural member for deflection.

(a). Stiffness of a beam hinged at both ends:



Stiffness of a beam hinged at both ends, k = $\frac{3EI}{l}$

(b). Stiffness of a beam hinged at near end and fixed at far end (k):



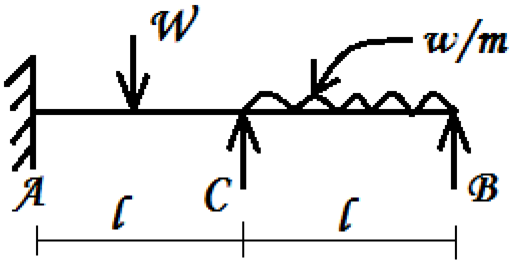
Stiffness of a beam hinged at near end and fixed at far end,

k = $\frac{4EI}{l}$

2). Relative stiffness:

The ratio of stiffness of various members meeting at a structural joint is known as “relative stiffness”.

Explanation:



For the continuous beam shown in fig, B is the joint where members BA & BC meet.

Stiffness of BA = $\frac{4EI}{l}$ (far end A is fixed)

Stiffness of BC = $\frac{3EI}{l}$

Relative stiffness

Stiffness of BA: Stiffness of BC

$\frac{4EI}{l}$: $\frac{3EI}{l}$

If the beam is made of same material, then relative stiffness

$=\frac{4EI}{l}$: $\frac{3EI}{l}$

(Or)

Dividing by 4,

Relative stiffness

$=\frac{I}{l} : \frac{3I}{4l}$

3). Distribution factor:

The ratio of stiffness of a member meeting at a structural joint to the sum of the stiffness of all members meeting at that joint is known as “distribution factor”.

For ex:

$D.F_{BA} = \frac{K_{BA}}{\sum K}$

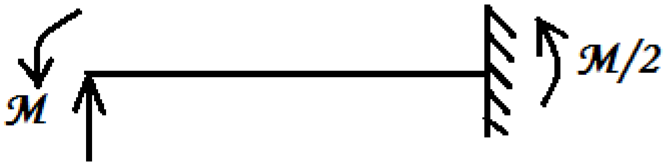
4). Distribution moment:

The moment shared by a member at a joint in proportion to its stiffness or in relation to its distribution factor (D.F) is known as “distribution moment”. It is also known as “balancing moment”.

5). Carryover moment:

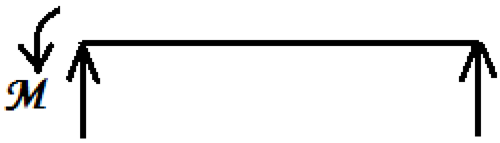
The moment produced at the far end of a beam due to application of a moment at the near end is called “carryover moment”.

i. When the far end is fixed:



The carry over moment is ½ the applied moment in the same direction.

ii. When the far end is hinged:



No carryover

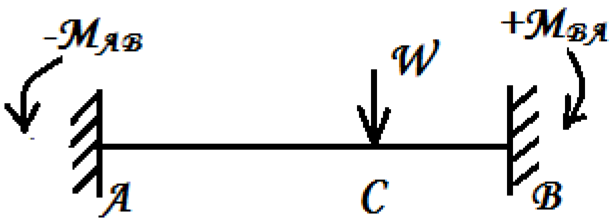
There is no carry over.When the far end is hinged.

6).Carry over factor:

The ratio of carry over moment at the far end to the applied moment at the near end is known as carry over factor.

Carry over factor $= \frac{Carry\ Over\ Moment}{Applied\ Moment}$

Sign convention:



The new sign convention different from conventional sign in followed in the process of moment – distribution method. After the analysis is over, the end moments are converted back to conventional bending moments by merely changing the sign to the left of each span.

Based on rotational sense, Clockwise moments are +ve, anticlockwise moments are –ve as shown in figure.

Moment distribution method procedure (Theory):

1) Assume all the supports (joints) are fixed.

2) Calculate the **fixed end moments** (FEM) due to external loads considering each span as a separate fixed beam.

3) Calculate the stiffness, relative stiffness and hence the **distribution factors** of members meeting at each intermediate joint.

Note:

i. D.F of a member at its fixed end is zero.
ii. D.F of a member at its hinged end is one.
iii. D.F of an overhanging member at its joint is zero.

4)Prepare a moment – distribution table in tabular form and enter D.F & F.E.M with proper signs.

5) Release each clamped support in succession. Distribute the unbalanced moment at each joint among the connecting members according to their distribution factor with a sign opposite to the unbalanced moment. This is called as 'Balancing the joint'.

6) Do **carry over**, one half of each distributed moment with the **same sign** to the farther end of each span.

Note:

i. Do carry over to farther fixed ends and intermediate supports only.
ii. Do not do carry over to farther simply supported end and farther overhanging ends.

This completes one cycle of moment distribution.

7) The carry over moments in step 6 cause new unbalanced moments. Hence perform distribution and carry over as explained in steps 5&6 to complete the second cycle.

8) Repeat the process of distribution and carryover until the carry over moments become zero or negligibly small.

Note:

i. The accuracy depends upon the no. of cycles.
ii. Generally 4 or 5 cycles will be sufficient as the unbalanced moments caused by carryover decrease rapidly.

9) Stop the process with distributor at the intermediate supports when the end supports are simply supported.

(Or)

With carry over to the fixed ends when the supports arefixed.

10) Arrive at the final moments at each joint by finding the algebraic sum of moments in each vertical column.

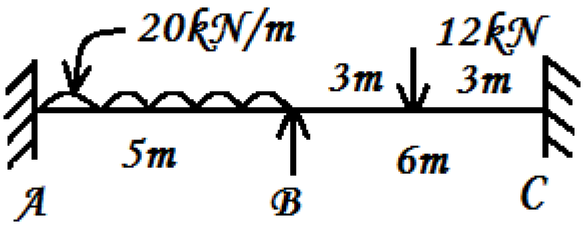
11) Change the signs of final moments to the left of the support to get conventional moments.

APPLICATION OF MOMENT – DISTRIBUTION TO TWO SPAN CONTINUOUS BEAMS

Worked examples

CONTINUOUS BEAM WITH FIXED ENDS

1) A continuous beam ABC is fixed at A & C. It is loaded as shown in fig. Calculate the support moments and draw SFD & BMD. Assume EI as constant.

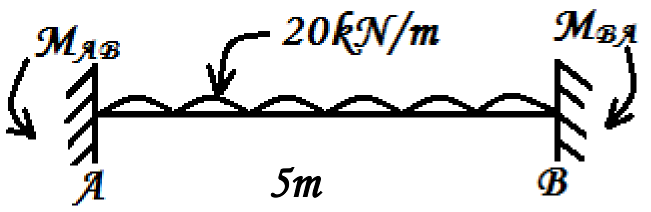


Step 1: Fixed end moments

Considering each span as fixed.

Span AB:

Adopting clockwise as +ve & anticlockwise as –ve.



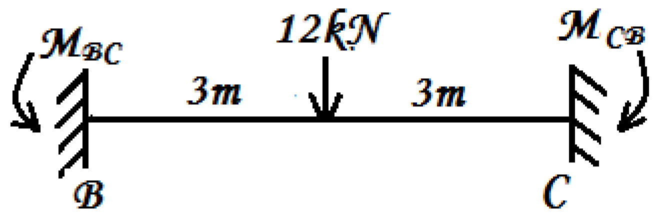
$M_{AB} = \frac{-wl^2}{12} = -\frac{20 \times 5^2}{12}$

$= -41.67$kNm

$M_{BA} = \frac{+wl^2}{12} = +\frac{20 \times 5^2}{12}$

$= +41.67$kNm

Span BC:



$M_{BC} = \frac{-wl}{8} = \frac{-12 \times 6}{8}$

$= -9$kNm

$M_{CB} = \frac{+wl}{8} = \frac{+12 \times 6}{8}$

$= +9$kNm

<u>Step 2: Distribution Factor</u>

(1) Joint: Only one joint: B

Joint	Member	Stiffness	Relative stiffness (k)	Σk	Distribution factor (D.F)
B	BA BC	$\frac{4EI}{l} = \frac{4EI}{5}$ (far end A is fixed) $\frac{4EI}{l} = \frac{4EI}{6}$ (far end C is fixed)	$K_{BA} : K_{BC}$ $\frac{4EI}{5} : \frac{4EI}{6}$ 6:5	$= K_{BA} + K_{BC}$ $= 6 + 5$ $= 11$	$D.F_{BA} = \frac{K_{BA}}{\Sigma k}$ $= 6/11$ $D.F_{BC} = \frac{K_{BC}}{\Sigma k}$ $= 5/11$

(2) Supports (A & C):

A & C are end supports and they are fixed supports. No distribution is done at the fixed ends and D.F at the fixed ends are 0.

Step 3: Moment distribution table

(support) A (joint) B (support) C

Member	AB	BA	BC	CB
Distribution factor (D.F)	0	6/11	5/11	0
Fixed end moments (FEM)	-41.67	+41.67	-9.00	+9.00
I Distribution at B	0.00	-17.82	-14.85	0.00
Carry Over (C.O) to A & C	-8.91			-7.43
Final moments (algebraic sum)	-50.58	+23.85	-23.85	+1.57
Conventional BM (change the sign at left of B & C)	-50.58	-23.85	-23.85	-1.57

Moment distribution process (explanation):

- A moment distribution table is prepared and FEM & DF are entered.
- A & C are fixed supports. Hence there is no distribution at A & C.
- At joint B,

Unbalanced moment = +41.67 – 9.00 = +32.67

∴ Balancing moment at B = - 32.67

I distribution to BA (acc to DF_{BA}) = $\frac{6}{11} \times -32.67$ = -17.82

I distribution to BC (acc to DF_{BC}) = $\frac{5}{11} \times -32.67$ = -14.85

These values are entered in second step at table. (acc – according)

- The far ends A & C are fixed. Hence one half of distributed moments with the same sign are carried over (C.O) to A & C.

C.O to A from B = -17.82 × ½ = - 8.91kNm

C.O to C from B = -14.85 × ½ = - 7.43kNm

These values are entered in the third step.

- Since there is no chance for further distribution or carry over, the process comes to a halt and the process is stopped.
- To get the final moments, all the moments at each support are algebraically added.
- To get back the conventional moments, the sign of final moments to the left of each support are changed. (Here +sign to – sign).

BMD:

Superpose the free BMD & fixed BMD.

Free BMD:

Span AB

Max free BM at D = $\frac{+wl^2}{8} = \frac{+20 \times 5^2}{8}$

= + 62.5kNm

Span BC

Max free BM at E = $\frac{+Wl}{4} = \frac{+12 \times 6}{4}$

= 18kNm

Fixed BMD:

M_A = - 50.58kNm

M_B = - 23.85kNm

M_C = - 1.57kNm

SFD:

Let V_A, V_B & V_C be the vertical support reactions.

(i). Taking moments about B & considering left of B.

$R_A \times 5 - M_{AB} - (20 \times 5 \times 5/2) + M_{BA} = 0$

$5R_A - 50.58 - 250 + 23.85 = 0$

$R_A = 55.35kN(\uparrow)$

(ii). Again taking moments about B & considering right of B.

$-R_C \times 6 + M_{CB} + 12 \times 3 - M_{BC} = 0$

$-6R_C + 1.57 + 36 - 23.85 = 0$

$R_C= 2.29kN(\uparrow)$

From $\Sigma_V = 0$

$R_A + R_B + R_C - 20 \times 5 - 12 = 0$

$55.35 + R_B + 2.29 - 100 - 12 = 0$

$R_B= 54.36kN$

Shear force (V_X):

$V_A = + R_A = + 55.35kN$

$V_B(L)= + R_A - 20 \times 5 = + 55.35 - 100$

$= -44.65kN$

$V_B(R) = +R_A - 20 \times 5 + R_B = +55.35 - 100 + 54.36$

$= +9.71kN$

$V_E(L) = V_B(R) = +9.71kN$

$V_E(R) = V_E(L) - 12 = +9.71 - 12$

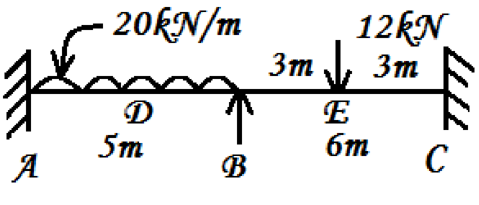
$= -2.29kN$

$V_C(L) = V_E(R) = -2.29kN$

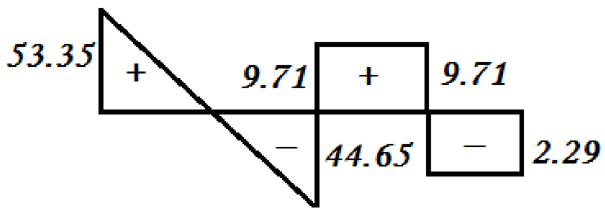
$V_C(R) = 0$

Complete SFD.

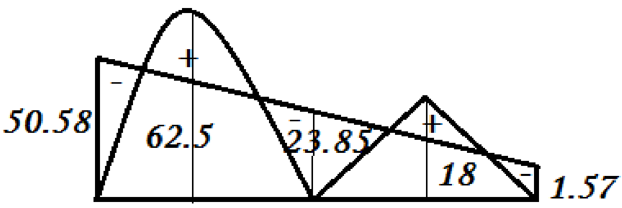
Results:



Loading diagram



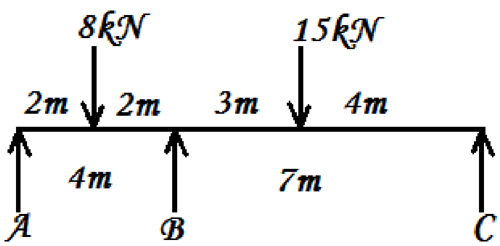
SFD



BMD

CONTINUOUS BEAM WITH SIMPLY – SUPPORTED ENDS

2) For the continuous beam ABC shown in fig, find the support moments by moment – distribution method. Draw BMD.

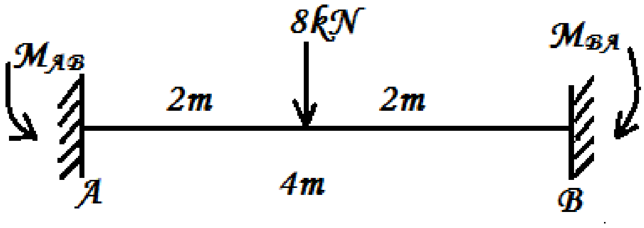


Solution:

Step 1: Fixed end moments

Considering each span as fixed.

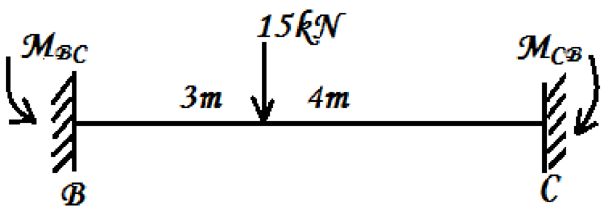
Span AB



$M_{AB} = \frac{-Wl}{8} = \frac{-8 \times 4}{8} = -4$kNm

$M_{BA} = \frac{+Wl}{8} = +4$ kNm

Span BC



$M_{BC} = \frac{-Wab^2}{l^2} = \frac{-15 \times 3 \times 4^2}{7^2}$

$= -14.69$kNm

$M_{CB} = \frac{+Wa^2b}{l^2} = \frac{+15 \times 3^2 \times 4}{7^2} = +11.02$kNm

Step 2: Distribution Factor (D.F)

(1). Joint: Only one joint B

Joint	Member	Stiffness	Relative stiffness (k)	Σk	Distribution factor (D.F)
B	BA BC	$K_{BA} = \frac{3EI}{l}$ $= \frac{3EI}{4}$ (since far end A is simply supported) $K_{BC} = \frac{3EI}{l}$ $= \frac{3EI}{7}$ (since far end C is simply supported)	$K_{BA} : K_{BC}$ $\frac{3EI}{4} : \frac{3EI}{7}$ 7:4	$= K_{BA} + K_{BC}$ = 7 + 4 = 11	$D.F_{BA} = \frac{K_{BA}}{\Sigma k}$ = 7/11 $D.F_{BC} = \frac{K_{BC}}{\Sigma k}$ = 4/11

(2). End supports (A & C)

A & C are simply supported ends. The distribution factor is 1.

∴ $D.F_{AB} = 1$; $D.F_{CB} = 1$

Step 3: Moment – distribution table

	(support) A	(joint) B		(support) C
Member	AB	BA	BC	CB
Distribution factor(D.F)	1	7/11	4/11	1
Fixed end moments (FEM) **Release simple supports A & C.**	-4.00 +4.00	+4.00 *C.O to B* 2.00	-14.69 -5.51	+11.02 *C.O to B* 11.02
Adjusted initial FEM **I distribution at B**	0.0	+6.00 +9.04	-20.20 +5.16	0.00
Final moments (algebraic sum)	0.00	+15.04	-15.04	0.00
Conventional moments	0.00	-15.04	-15.04	0.00

Moment distribution process (Explanation):

- A moment distribution table is prepared and FEMs & DFs are entered.
- A & C are simply supported ends. The BM at A & C should be 0. Hence A & C are released by applying balancing moments of 4.00kNm at A and -11.02kNm at C and half their amounts are carried over with the same sign to B and thus the FEMs are adjusted and entered in the third row.
- Now the joint B is unbalanced

Unbalanced moment at B = +6.00 – 20.20 = -14.20kNm

∴ Balancing moment at B= + 14.20kNm

I distribution to BA (acc to DF_{BA}) = $+14.20 \times \frac{7}{11}$ = +9.03 kNm

I distribution to BC (acc to DF_{BC}) = $+14.20 \times \frac{4}{11}$ = +5.16 kNm

- Since there is no chance for further distribution & C.O, the process is stopped.
- To get the final moments, the moments at each support are algebraically added.
- To get back the conventional moments the sign of final moments to the left of each support are changed. (Here +sign to –sign).

BMD:

(1). Free BMD

Consider each span as simply supported.

Span AB

$$M_{max} = \frac{+Wl}{4} = \frac{+8 \times 4}{4} = +8\text{kNm}$$

Span BC

$$M_{max} = \frac{+Wab}{l} = \frac{+15 \times 3 \times 4}{7} = +25.71kNm$$

Complete free BMD

(2). Fixed BMD

$M_A = 0$

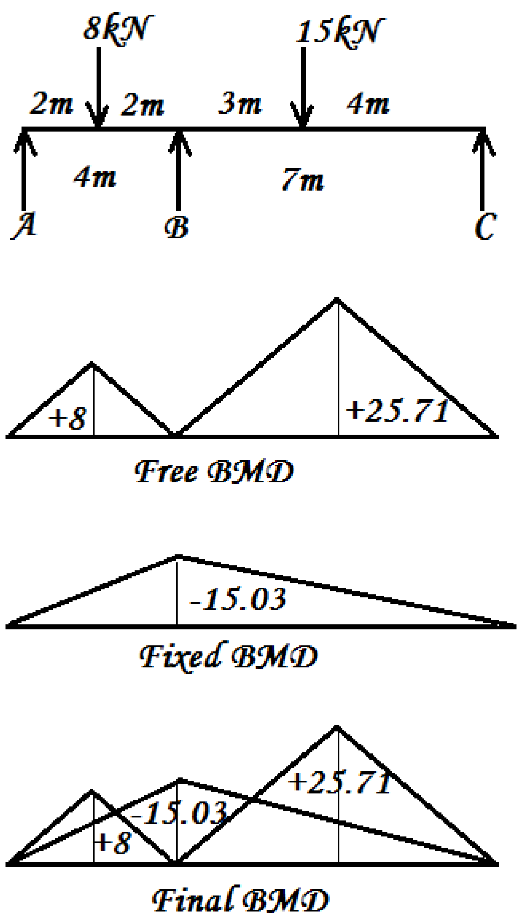
$M_B = -15.04kNm$

$M_C = 0$

Complete fixed BMD.

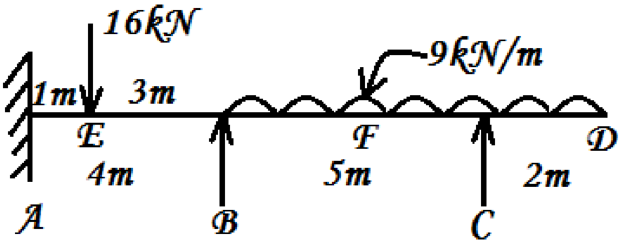
Superpose fixed BMD over free BMD to get BMD.

Complete final BMD



CONTINUOUS BEAMS WITH OVERHANGS

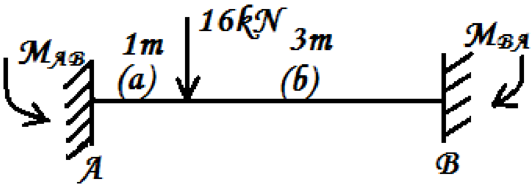
1) A continuous beam ABCD is fixed of A, simply supported at B, C & free at D. It is loaded as shown in fig. EI is constant throughout. Calculate the support moments and draw SFD & BMD.



Loading diagram

Step 1: Fixed end moments

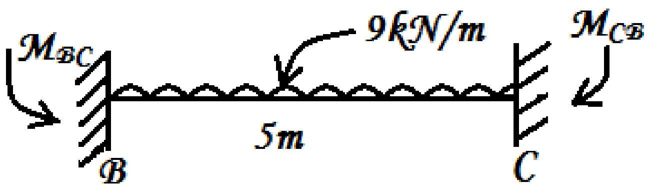
Span AB



$M_{AB} = \frac{-Wab^2}{l^2} = \frac{-16 \times 1 \times 3^2}{4^2} = -9$ kNm

$M_{BA} = \frac{+Wa^2b}{l^2} = \frac{+16 \times 1^2 \times 3}{4^2} = +3$ kNm

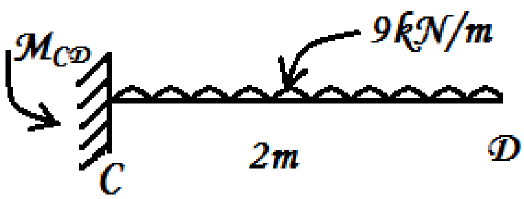
Span BC



$M_{BC} = \frac{-wl^2}{12} = \frac{-9 \times 5^2}{12} = -18.75$ kNm

$M_{CB} = \frac{+wl^2}{12} = +18.75$ kNm

Span CD



It behaves like a cantilever.

$M_{CD} = \frac{-wl^2}{2} = \frac{-9 \times 2^2}{2} = -18$ kNm

$M_{DC} = 0$ (free end)

Step 2: Distribution factor (D.F)

Joint B & C.

Joint	Member	Stiffness	Relative stiffness (k)	Σk	Distribution factor (D.F)
B	BA BC	$K_{BA} = \frac{4EI}{l} = \frac{4EI}{4}$ (since far end A is fixed) $K_{BC} = \frac{3EI}{l} = \frac{3EI}{5}$ (since far end C is simply supported)	$K_{BA} : K_{BC}$ $\frac{4EI}{4} : \frac{3EI}{5}$ 1: 3/5 5:3	$= K_{BA} + K_{BC}$ = 5 + 3 = 8	$D.F_{BA} = \frac{K_{BA}}{\Sigma k}$ = 5/8 $D.F_{BC} = \frac{K_{BC}}{\Sigma k}$ = 3/8
C	CB CD	1 0	1:0	1	$D.F_{CB} = 1$ $D.F_{CD} = 0$

Step 3: Moment distribution table

A B C D

Member	AB	BA	BC	CB	CD	DC
Distribution factors	0	5/8	3/8	1	0	
Fixed end moments (FEM)	-9.00	+3.00	-18.75	+18.75	-18.00	
Release C & C.O to B			-0.38	-0.75		
Initial/ Adjusted FEM **I distribution at B**	-9.00	+3.00 +10.08	-19.13 +6.05			
Carry over (To A from B)	+5.04	-	-		-	-
Final moments (Σ) (algebraic sum)	-3.96	+13.08	-13.08	+18.00	-18.00	0
Conventional moments	-3.96	-13.08	-13.08	-18.00	-18.00	0

Σ

SFD:

Left of B

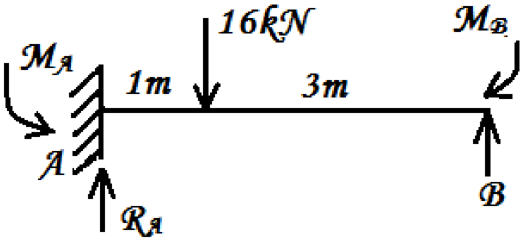
Considering left of B and taking moments about B.

$M_A + R_A \times 4 - 16 \times 3 + M_B = 0$

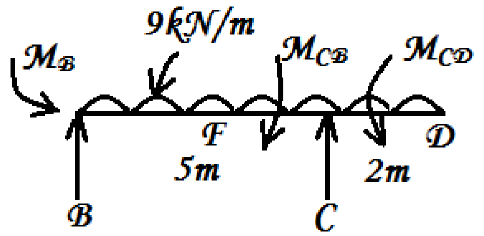
$-3.96 + R_A \times 4 - 48 + 13.08 = 0$

$4R_A = 38.88$

$R_A = 9.72kN(\uparrow)$



Considering right of B



Taking moments about B

$-M_B + 9 \times 7 \times 7/2 - R_C \times 5 = 0$

$-13.08 + 220.5 - 5R_C = 0$

$R_C = 41.48kN(\uparrow)$

From $\Sigma_V = 0$

$R_A + R_B + R_C = 16 + 9 \times 7$

$9.72 + R_B + 41.48 = 79$

$R_B = 27.8kN$

Shear force (V_X):

$V_A(L) = 0$

$V_A(R) = + R_A = + 9.72kN$

$V_E(L) = V_A(R) = + 9.72kN$

$V_E(R) = V_E(L) - 16 = + 9.72 - 16$

$= -6.28kN$

$V_B(L) = V_E(R) = -6.28kN$

$V_B(R) = V_E(R) + R_B = -6.28 + 27.8$

$= +21.52kN$

$V_C(L) = V_B(R) - 9 \times 5 = +21.52 - 45$

$= -23.48kN$

$V_C(R) = V_C(L) + R_C = -23.48 + 41.48$

$= +17.7kN$

$V_D \quad = 0$

Complete SFD.

BMD:

Span AB – free BMD (sagging BM)

Max free BM for span AB

$$= \frac{+Wab}{l} = \frac{+16 \times 1 \times 3}{4}$$

$= +12kNm$

Span BC

Max free BM for span BC

$$= \frac{+wl^2}{8} = \frac{+9 \times 5^2}{8}$$

$= +28.13kNm$

Span CD

No free BM since it behaves like a cantilever.

Fixed BMD (hogging BM)

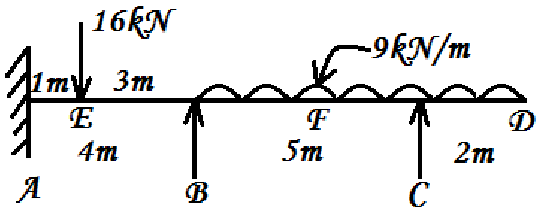
At support A $M_A = -3.97$

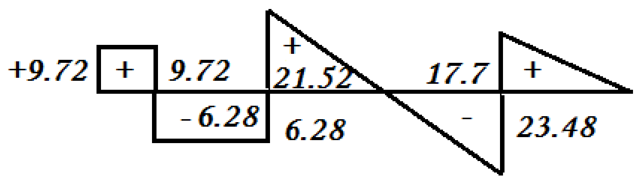
At support B $M_B = -13.06$

At support C $M_C = -18.00$

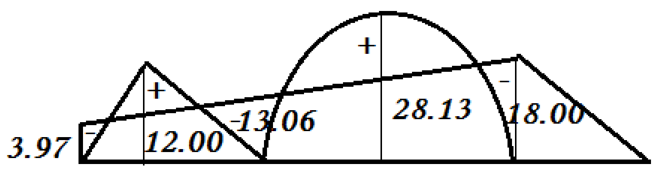
Complete BMD by superposing fixed BMD over free BMD.

Final BMD:





SFD

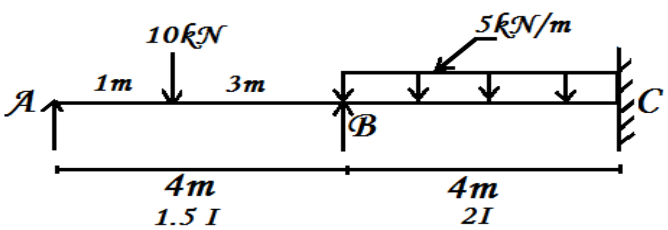


BMD

TWO – SPAN CONTINUOUS BEAMS

(**EI** not constant: **E**Constant**, I** Varying)

4. Find the support moments for the continuous beam loaded as shown in fig.



Solution:

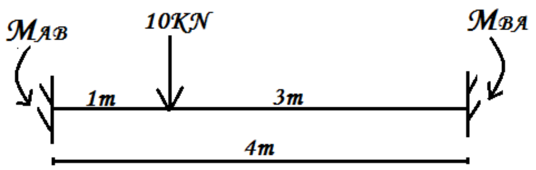
Step 1: Fixed End Moments:

Span AB:

M_{AB} $= -\frac{Wab^2}{l^2}$

$= -\frac{10 \times 1 \times (3)^2}{4^2}$

= -5.63kNm.

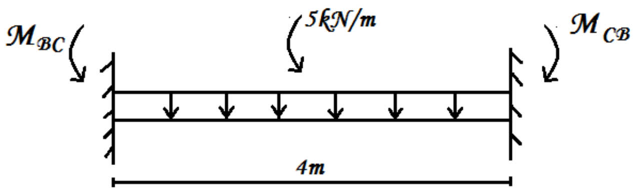


$M_{BA} \quad = +\dfrac{Wa^2b}{l^2}$

$= +\dfrac{10 \times 1^2 \times 3}{(4)^2}$

$= + 1.88$kNm.

Span BC:



$M_{BC} \quad = -\dfrac{wl^2}{12}$

$= -\dfrac{5 \times (4)^2}{12}$

$= -6.67$kNm.

$M_{CB} \quad = +\dfrac{Wl^2}{12}$

$= +6.67$kNm.

Step 2: Distribution Factors (DF):

1. **Joints (Only one joint B)**

Joint	**Member**	**Stiffness**	**Relative Stiffness**	**$\sum$k**	**D.F = k/$\sum$k**
B	BA	$K_{BA} = \dfrac{3EI}{l}$ (*since* A is simply supported) $= \dfrac{3E(1.5I)}{l}$ $= \dfrac{4.5\,EI}{4}$.	$K_{BA}:K_{BC}$ $= \dfrac{4.5EI}{4} : \dfrac{8EI}{4}$ 4.5 : 8 (i.e.) 9 : 16	9+16 = 25	$DF_{BA} = \dfrac{k_{BA}}{\sum k}$ $= \dfrac{9}{25}$
	BC	$K_{BC} = \dfrac{4EI}{l}$ (*since* C is fixed) $= \dfrac{4E(2I)}{l}$ $= \dfrac{8EI}{4}$.			$DF_{BC} = \dfrac{k_{BC}}{\sum k}$ $= \dfrac{16}{25}$

2. **End supports:**
   i. A is simply supported ∴ $DF_{AB} = 1$
   ii. C is fixed ∴ $DF_{CB} = 0$

Step 3: Moment Distribution Table:

A B C

Member	AB	BA	BC	CB
Distribution Factor	1	9/25	16/25	0
FEM	-5.63	+1.88	-6.67	+6.67
Release A & C.O to B	+5.63	2.82		
Initial / Adjusted FEM	0.00	+4.70	-6.67	+6.67
I – distribution at B		+0.70	1.27	
C.O to C				0.64
Final moments ∑	0.00	+5.40	-5.40	+7.31
Conventional moments	0.00	-5.40	-5.40	-7.31

<u>Moment Distribution Process :</u>(Explanation)

- The FEM & DF are calculated and are entered in the moment distribution table.
- Since A is a simple support, it is released to make the moment zero. Half the moments is carried over to B. thus the FEMs are initially adjusted.
- Since B is unbalanced, it is balanced and distribution is done to BA & BC according to DF.
- One half of the balanced moment is carried over to C from B.
- Since there is no chance for further distribution or C.O, the process is stopped.
- To get the final moments algebraic sum from adjusted FEM is done.
- To get back the conventional moments the sign of moment is changed at the left of each support.

BMD

1. **Free BMD**

   Maximum free BM for span AB =+ $\frac{Wab}{l}$

   $= +\frac{10\times1\times3}{4}$

   = +7.5kNm

Maximum free BM for span BC = +$\frac{wl^2}{8}$

$= +\frac{5\times4^2}{8}$

= +10kNm

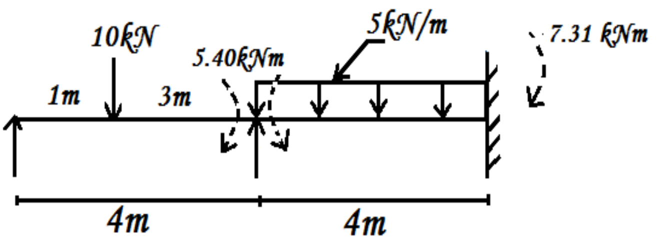
2. **Fixed BMD**

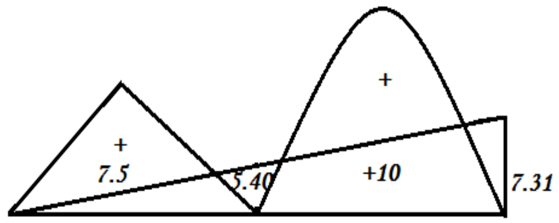
$M_A = 0$

$M_B = -5.40kNm$

$M_c = -7.31kNm$

3. **Final BMD**



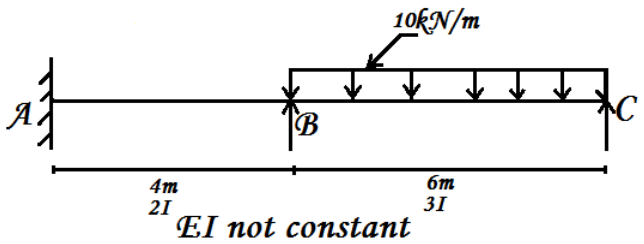


Final BMD

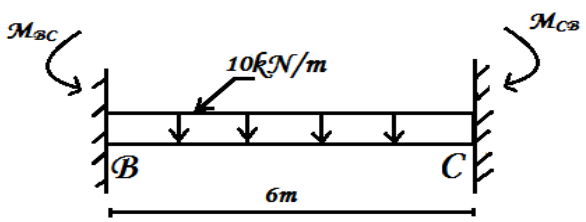
TWO - SPAN CONTINUOUS BEAM

(**EI** not constant, **I** Varying)

5. Find the support moments for the two-span continuous beam ABC shown in fig. Also draw BMD.



Step 1: Fixed End Moments:



Span AB:

No loading in span AB

$\therefore M_{AB} = M_{BA} = 0$

Span BC:

$$M_{BC} = -\frac{wl^2}{12}$$

$$= -\frac{10\times6^2}{12}$$

$= \quad -30kNm$

$M_{CB} = +\frac{wl^2}{12}$

$= +\frac{10 \times 6^2}{12}$

$= \quad +30kNm$

Step 2: Distribution Factors (DF):

i. Joints (only one joint B)

Joint	Member	Stiffness	Relative Stiffness	∑k	D.F = k/∑k
B	BA	$K_{BA} = \frac{4EI}{l} = \frac{4E(2I)}{4} = \frac{8EI}{4}$.	$K_{BA} : K_{BC}$ $= \frac{8EI}{4} : \frac{9EI}{6}$	4+3 = 7	$DF_{BA} = \frac{k_{BA}}{\sum k} = \frac{4}{7}$
	BC	$K_{BC} = \frac{3EI}{l} = \frac{3E(3I)}{6} = \frac{9EI}{6}$.	$\frac{8}{4} : \frac{9}{6}$ 24 : 18 (i.e.) 4:3		$DF_{BC} = \frac{k_{BC}}{\sum k} = \frac{3}{7}$

ii. Supports:

1. A is fixed ∴ $DF_{AB} = 0$.
2. C is simply supported ∴ $DF_{CB} = 1$.

Step 3: Moment distribution Table:

A B C

Member	AB	BA	BC	CB
Distribution Factor	0	4/7	3/7	1
FEM	0.00	0.00	-30.00	+30.00
Release C & C.O to B			-15.00	-30.00
Initial / Adjusted FEM	0.00	0.00	-45.00	0.00
I – distribution		+25.72	+19.28	
C.O to A	+12.86			
Final moments ∑	+12.86	+25.71	-25.71	0.00
Conventional moments	-12.86	-25.71	-25.71	0.00

∑

Moment Distribution Process: (Explanation)

- A moment distribution table is prepared. The FEM & DF are entered.
- Since C is a simple – support, to make the moment zero, it is released by balancing it and half the balanced moment is carried over to B and thus the FEM are adjusted.
- Now I distribution is done at B.
- Since 'A' is a fixed support, half the moment is carried over to A from B.
- Since there is no chance for further distribution & C.O, the process is stopped.
- To get final moments, add the moment algebraically from adjusted FEM.
- To get back the conventional moments, the signs in the left support moments are changed.

Step 4: BMD

1. **Free BMD: (sagging)**

Maximum free BM for span AB = 0 (*since* no loading)

Maximum free BM for span BC $=+\frac{wl^2}{8}$

$$= +\frac{10\times6^2}{8}$$

= +45kNm.

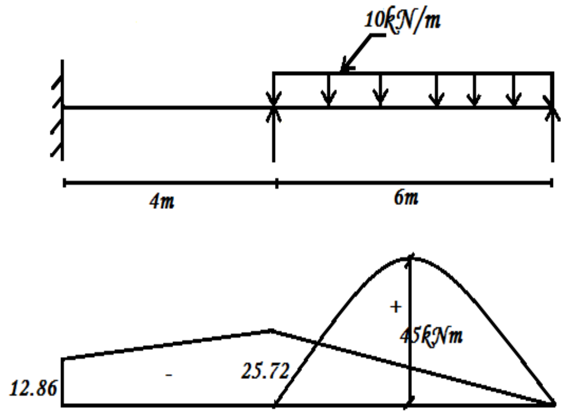
2. **Fixed BMD: (hogging)**

M_A = - 12.86kNm.

M_B= - 25.72kNm.

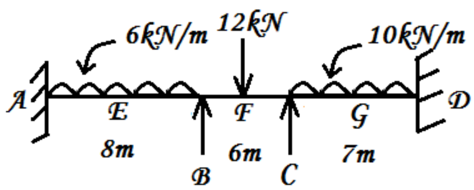
M_C = - 0kNm.

Final BMD:



THREE SPAN CONTINUOUS BEAMS

6) A continuous beam ABCD is fixed at A & D. It is loaded as shown in fig. Calculate the support moments by moment distribution method. Sketch BMD.

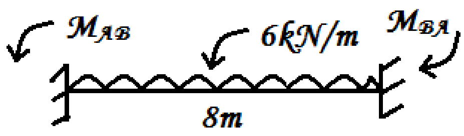


Solution:

Step 1: Fixed end moments (FEM)

Considering each span as fixed.

Span AB



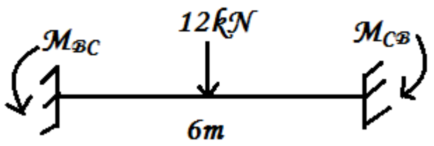
$$M_{AB} = \frac{-wl^2}{12} = \frac{-6 \times 8^2}{12}$$

$= -32kNm$

$$M_{BA} = \frac{+wl^2}{12} = \frac{+6 \times 8^2}{12}$$

$= +32kNm$

Span BC

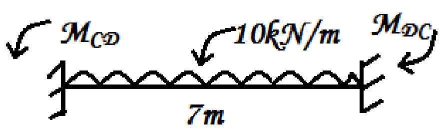


$$M_{BC} = \frac{-Wl}{8} = \frac{-12 \times 6}{8}$$

$= -9kNm$

$$M_{CB} = \frac{+Wl}{8} = +9kNm$$

Span CD



$$M_{CD} = \frac{-wl^2}{12} = \frac{-10 \times 7^2}{12}$$

$= -40.83kNm$

$$M_{DC} = \frac{+wl^2}{12} = \frac{+10 \times 7^2}{12}$$

$= +40.83kNm$

Step 2: Distribution factors (D.F)

(1). Joints B & C

Joint	Member	Stiffness	Relative stiffness (k)	Σk	Distribution factor (D.F)
B	BA BC	$K_{BA} = \frac{4EI}{l} = \frac{4EI}{8}$ $K_{BC} = \frac{4EI}{l} = \frac{4EI}{6}$	$K_{BA} : K_{BC}$ $\frac{4EI}{8} : \frac{4EI}{6}$ 6: 8	$= K_{BA} + K_{BC}$ = 6 + 8 = 14	$D.F_{BA} = \frac{K_{BA}}{\Sigma k}$ = 3/7 $D.F_{BC} = \frac{K_{BC}}{\Sigma k}$ = 4/7
C	CB CD	$K_{CB} = \frac{4EI}{l} = \frac{4EI}{6}$ $K_{CD} = \frac{4EI}{l} = \frac{4EI}{7}$	$K_{CB} : K_{CD}$ $\frac{4EI}{6} : \frac{4EI}{7}$ 7:6	7 + 6 13	$D.F_{CB}$ = 7/13 $D.F_{CD}$= 6/13

2) End supports (C, D):

End supports C & D are fixed, ∴ no distribution is done at C & D

∴ $DF_{AB} = 0$; $DF_{DC} = 0$

Step 3: Moment distribution table:

A B C D

Member	AB	BA	BC	CB	CD	DC
Distribution factors (DF)	0	3/7	4/7	7/13	6/13	0
Fixed end moments (FEM)	-32.00	+32.00	-9.00	+9.00	-40.83	+40.83
I distribution		-9.86	-13.14	+17.14	+14.69	
Carry over(C.O)	-4.93		+8.57	-6.57		+7.35
II distribution		-3.67	-4.90	+3.54	+3.03	
C.O	-1.84		+1.77	-2.45		+1.52
III distribution		-0.76	-1.01	+1.32	+1.13	
C.O	-0.38		+0.66	-0.51		+0.57
IV distribution		-0.28	-0.38	+0.75	-0.24	
Final moments (Σ) (algebraic sum)	-39.15	+17.43	-17.43	+22.22	-22.22	+50.27
Conventional moments	-39.15	-17.43	-17.43	-22.22	-22.22	-50.27

Moment distribution process (Explanation):

- The FEM and DF are entered in the table.
- The distribution and carry over process is continued upto IV distribution.
- Since the C.O moments after III distribution is very small, the process is stopped up to IV distribution. (Normally four distributions will be sufficient).

BMD:

1). Free BMD

Span AB

Max free BM at E = $\frac{+wl^2}{8} = \frac{+6 \times 8^2}{8}$

= + 48kNm

Span BC

Max free BM at F = $\frac{+Wl}{4} = \frac{+12 \times 6}{4}$

= + 18kNm

Span CD

Max free BM at G = $\frac{+wl^2}{8} = \frac{+10 \times 7^2}{8}$

= + 61.25kNm

2). Fixed BMD

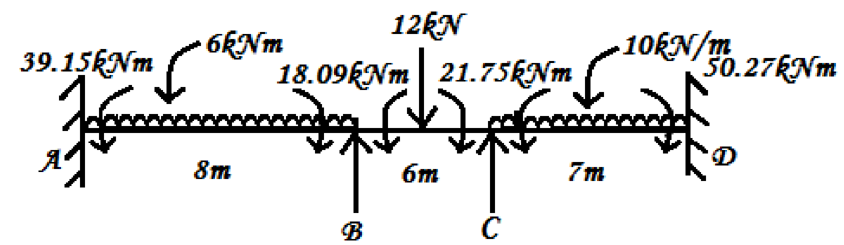
M_A = - 39.15kNm

M_B = - 17.43kNm

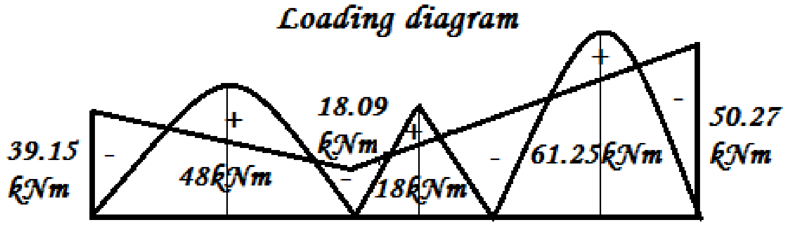
M_C = - 22.22kNm

M_D = -50.27kNm

Superpose fixed BMD over free BMD to get BMD.



Loading diagram



BMD

Results:

HIGH LIGHTS

1. a) Stiffness factor (k)

i. When far end is fixed stiffness factor $= \frac{4EI}{l}$

ii. When the far end is hinged stiffness factor $= \frac{3EI}{l}$

b) Relative stiffness

i. When the far end is fixed relative stiffness $= \frac{I}{l}$

ii. When the far end is hinged relative stiffness $= \frac{3I}{4l}$

c) Carry over moment

Carry over moment = carry over factor × applied moment.

2. Carry over factor

Carry over factor (COF) = $\frac{\text{Carry over moment}}{\text{Applied moment}}$

a. When the far end is fixed, COF = ½

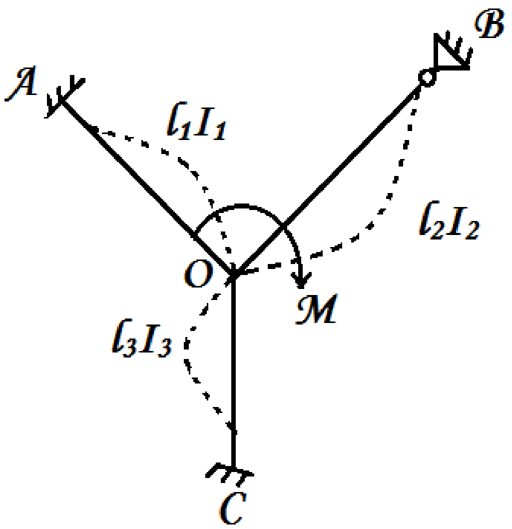
b. When the far end is hinged, COF = 0

Distribution Moment

Distribution moment = distribution factor ×applied moment.

Distribution factor (DF)

Distribution factor = $\frac{\text{Stiffness Factor of any member}}{\text{Total stiffness}}$



E.g.

In figure,

DF_{OA} = Distribution Factor for member OA

$DF_{OA} = \frac{K_{OA}}{\Sigma K_O}$

III^{ly} $DF_{OB} = \frac{K_{OB}}{\Sigma K_O}$

$DF_{OC} = \frac{K_{OC}}{\Sigma K_O}$

QUESTIONS

Two mark questions:

1) Where the (-ve) moment is maximum in a two span continuous beam having simple supports at the ends?
2) Define stiffness factor.
3) Define distribution factor.
4) Where the hogging (-ve) moment is maximum in a two span continuous beam, having simple supports at the ends?
5) Give examples of indeterminate beams.
6) State any two methods of analysis of indeterminate structures.
7) Define Distribution factor and Distribution moment.
8) What is Carry over factor?

Three mark questions:

1) A three span continuous beam with hinged ends carries UDL on its interior panel only. Draw the shapes of the SF and BM diagrams with proper signs (values need not be mentioned).
2) How do you analyse a continuous beam by Hardy cross method?
3) Prove that the stiffness of a simply supported beam of uniform cross section is 3EI/l.
4) Derive and expression for the stiffness of a beam when it is simply supported at both the ends.
5) Derive the expression for the stiffness of a beam when it is fixed at one end and freely supported at the other end.

Ten mark questions:

1) A continuous beam of ABC, simply supported at A and C, carries an UDL of 20kN/m on AB = 6m and carries a central point load of 120kN on BC = 6m. Take EI as constant. Draw SFD and BMD by moment distribution method.
2) A continuous beam of ABC, fixed at A& C, carries a point load of 5kN at 4m from A on the span AB = 6m and carries an UDL of 1.5kN/m on the span BC = 4m. Take EI as constant. Draw SFD and BMD by moment distribution method.
3) Compute the support moments by Hardy cross method for the two span continuous beam ABC with simply supported ends. All carries an UDL of 20kN/m. BC carries a point load of 90kN at 2m from B. $I_{BA} = I$, $I_{BC} = 2I$.
4) A beam ABCD, 9m long is simply supported at A, B and C, such that the span AB is 3m, span BC is 4.5m and the overhanging CD is 1.5m. It carries an UDL of 30kN/m in span AB and a point load of 10kN at the free end. The MI of the span AB is I and in span BC is 2I. Compute the support moments by moment distribution method.

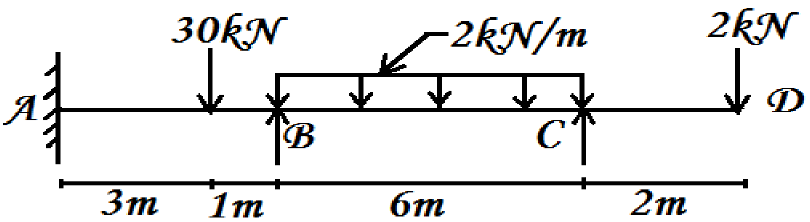
5) A continuous beam ABCD has three equal spans AB = BC = CD = 4m. It is simply supported at the ends A & D. AB carries an UDL of intensity 20kN/m; BC carries a central point load of 40kN; span CD carries an eccentric point load of 30kN at 1m from D. The flexural rigidity, EI is constant. Analyse the beam by Moment Distribution method (3 cycles sufficient) and draw the SF & BM diagram.

6) A continuous beam of ABC, fixed at A & C, carries an UDL of 30kN/m on AB = 6m and carries a central point load of 180 kN on BC = 6m. Take EI as constant. Draw SFD and BMD by moment distribution method.

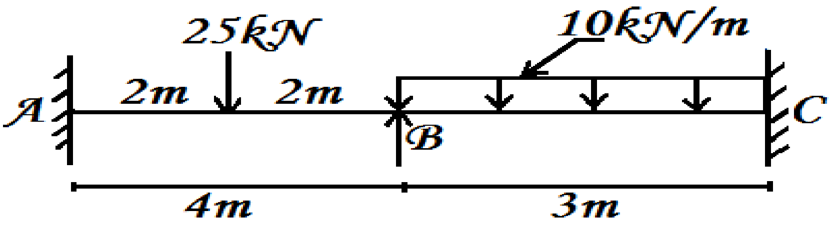
7) A continuous beam of ABC, fixed at A and C, carries a point load of 5kN at 4m from A on the span AB = 6m and carries an UDL of 1.5kN/m on the span BC = 4m. Take EI as constant. Draw SFD and BMD by moment distribution method.

8) A two span continuous beam ABC is fixed at support A and simply supported at support C. AB = 8m; BC = 4m. Span AB carries an UDL of 16kN/m; BC carries a central point load of 80kN. $I_{AB} = 1.5\ I_{BC}$. Analyse the beam by moment distribution method and draw BMD.

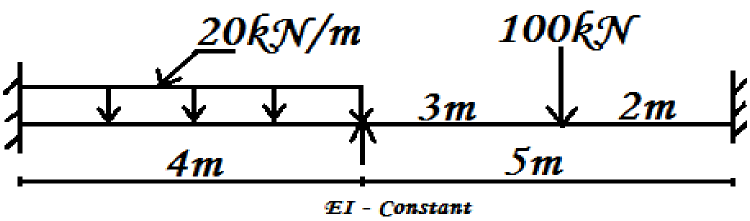
9) Analyse the continuous beam shown in fig. By moment distribution method. Find the support moments and draw the BMD. Assume EI as constant.



10) Analyse the continuous beam shown in fig. By moment distribution method. Draw the BMD. EI is constant.



11) Using Hardy cross method, determine the support moments and draw the BMD for the continuous beam shown in fig.



3.2 PORTAL FRAMES MOMENT DISTRIBUTION METHOD

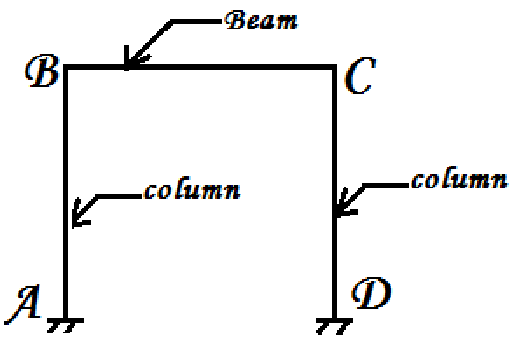
Frames:

A structure built – up of several members (beam over columns) joined together by rigid at their ends to support the external loads (vertical and horizontal) is called a frame. It is an indeterminate structure.

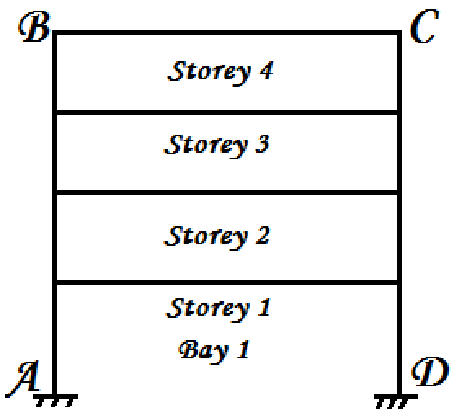
Types of frames:

Frames may be classified as

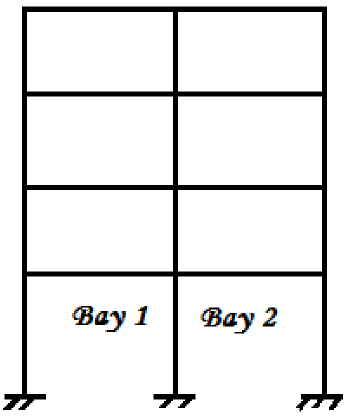
1. Based on bays
   - (i) Single bay single storey frame (Portal frame)
   - (ii) Single bay multi – storey frames
   - (iii) Frames multi bay frame
2. Based on storey
   - (i) Single storey frame
   - (ii) Multi storey frame



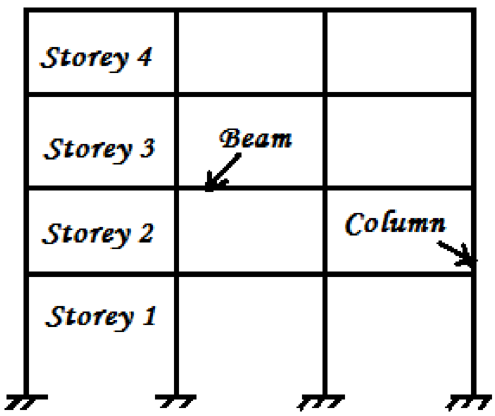
a) Single bay frame



b) Single bay multistorey frame



c)Two bay multistorey frame



d) Multistorey frames

a) Single bay frame:

When a frame consists of single bay it is called single bay frame. It may be of single storey or multi-storey as shown in fig.

When a frame consists of single bay with multi-storey as shown in fig.

b) Multi bay frame:

When a frame consists of two or more bays it is called multi bay frame. It may be of multi bay multi-storey frame as shown in fig.

Portal frame:

A frame consisting of beam resting on columns with rigid joints is known as portal frame.

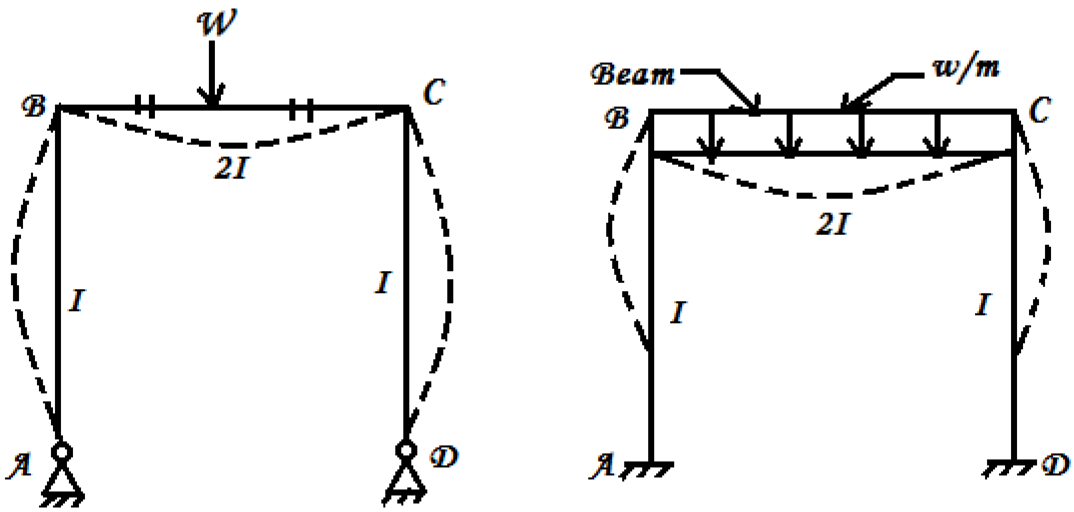
Classification of portal frame:

Portal frame classified as

a) Symmetrical portal frame
b) Unsymmetrical portal frame
c) Sway type portal frame
d) Non – sway type portal frame

a) Symmetrical portal frame:

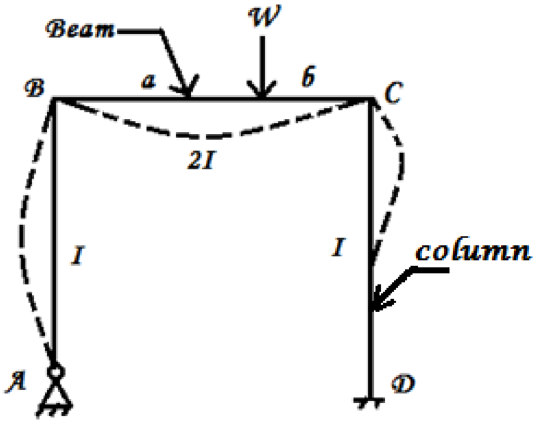
A portal frame, in which both the columns are having of the same length, geometry shape, similar end conditions, moments of inertia, modulus of elasticity and subjected to symmetrical loading as shown in fig. is called symmetrical portal frame.

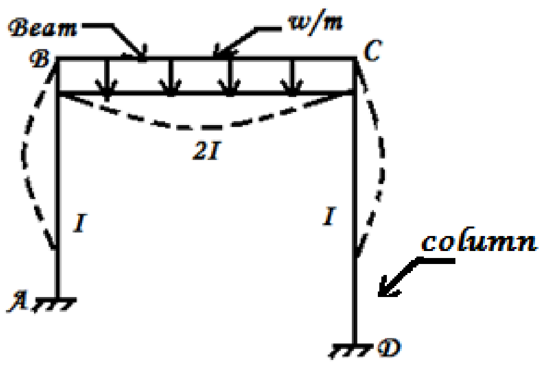


a) Symmetrical portal frame

b) Unsymmetrical portal frame:

A portal frame, in which both the columns are not having of the same length, geometry shape, similar end conditions, moments of inertia, modulus of elasticity and subjected to unsymmetrical loading as shown in fig. is called unsymmetrical portal frame.





b) Unsymmetrical Portal frame

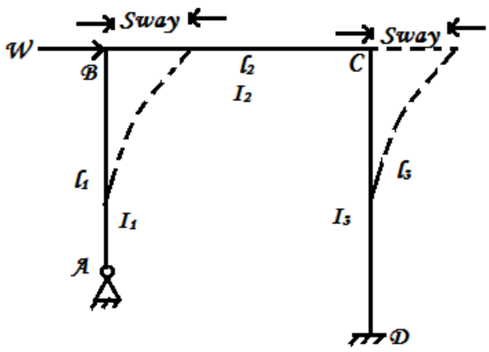
c) Sway type portal frame:

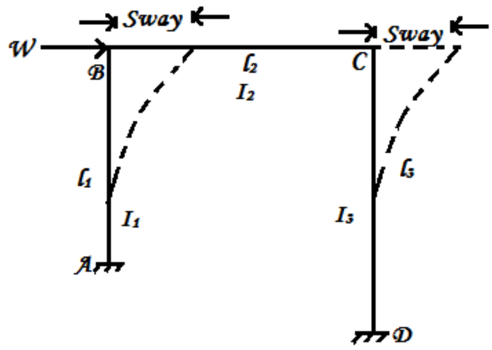
In case of unsymmetrical portal frame, the frame deflects horizontally. The frame having horizontal deflection is called sway type portal frame. In this case sway moments are considered. If the symmetrical portal frame is loaded asymmetrically sway moments are also considered.

These may be classified into

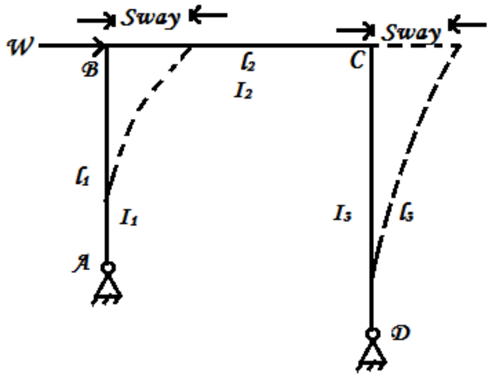
i) Pure sway frame

ii) General sway frame





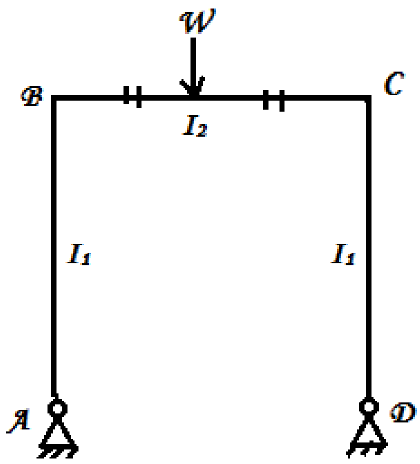
Portal frame with sway

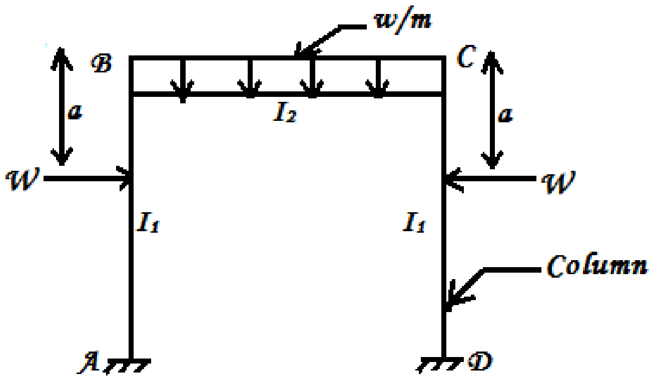


d) Non – sway type portal frames:

In case of symmetrical portal frame with symmetrical load horizontal deflection will not occur, this frame is called non – sway type portal frame.

This type of portal frame are analysed as in the case of continuous beams.



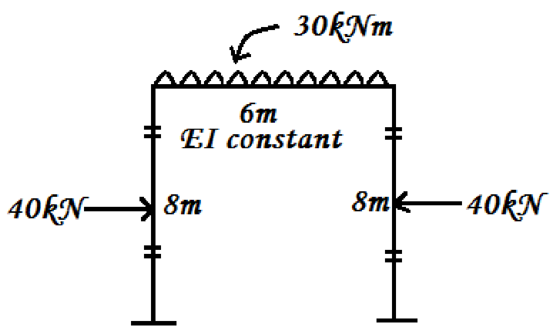


d) Non Sway Type Portal frame

Analysis of symmetrical (non – sway) portal frames by moment distribution method

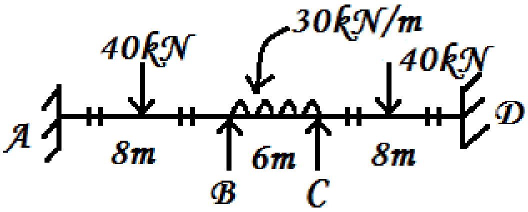
Problems: **WORKED EXAMPLE 1**

A portal frame ABCD is shown in fig. AB is loaded with central point load of 40kN at 8m and BC is loaded with an u.d.l of 30 kN/m throughout and CD is loaded with the same central point load of 40kN at 8m. If EI is constant throughout. Calculate the bending moments in the frame and draw the BMD.



Solution:

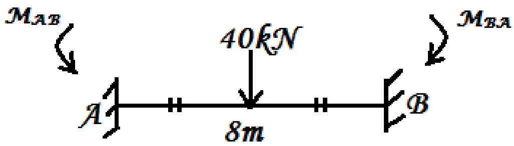
Stretch out the frame horizontally. The frame is equivalent to a continuous beam as shown in fig.



Step 1: Fixed end moments (FEM)

Consider each span as fixed:

Span AB



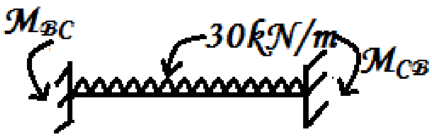
$M_{AB} = \frac{-Wl}{8} = \frac{-40 \times 8}{8}$

$= -40kNm$

$M_{BA} = \frac{+Wl}{8} = \frac{+40 \times 8}{8}$

$= +40kNm$

Span BC



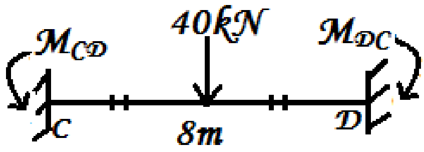
$M_{BC} = \frac{-wl^2}{12} = \frac{-30 \times 6^2}{12}$

$= -90kNm$

$M_{CB} = \frac{+wl^2}{12} = \frac{+30 \times 6^2}{12}$

$= +90kNm$

Span CD



$M_{CD} = \frac{-Wl}{8} = \frac{-40 \times 8}{8}$

$= -40kNm$

$M_{DC} = \frac{+Wl}{8} = \frac{+40 \times 8}{8}$

$= +40kNm$

Step 2: Distribution factors (DF)

(1) joints (B, C)

Joint	Member	Stiffness	Relative stiffness (k)	Σk	Distribution factor (D.F)
B	BA BC	$K_{BA} = \frac{4EI}{l} = \frac{4EI}{8}$ $K_{BC} = \frac{4EI}{l} = \frac{4EI}{6}$	$K_{BA} : K_{BC}$ $\frac{4EI}{8} : \frac{4EI}{6}$ 6: 8	$= K_{BA} + K_{BC}$ = 6 + 8 = 14	$D.F_{BA} = \frac{K_{BA}}{\Sigma k}$ = 3/7 $D.F_{BC} = \frac{K_{BC}}{\Sigma k}$ = 4/7
C	CB CD	$K_{CB} = \frac{4EI}{l} = \frac{4EI}{6}$ $K_{CD} = \frac{4EI}{l} = \frac{4EI}{8}$	$K_{CB} : K_{CD}$ $\frac{4EI}{6} : \frac{4EI}{8}$ 8:6	8 + 6 14	$D.F_{CB}$ = 8/14 = 4/7 $D.F_{CD}$= 6/14 = 3/7

(2)End supports (A & D)

End supports A & D are fixed. No distribution is done at A & D.

∴ $DF_{AB} = 0$

$DF_{DC} = 0$

Step 3: Moment distribution table

	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
Distribution factors(D.F)	0	3/7	4/7	4/7	3/7	0
(FEM)	-40.00	+40.00	-90.00	+90.00	-40.00	+40.00
I distribution		+21.43	+28.57	28.57	-21.43	
Carry over (C.O)	+10.72		-14.29	+14.29		10.72
II distribution		+6.12	+8.17	-8.17	-6.12	
C.O	+3.06		-4.09	+4.09		-3.06
III distribution		+1.75	+2.34	-2.34	-1.75	
C.O	+0.88		-1.17	+1.17		-0.88
IV distribution		+0.50	+0.67	-0.67	-0.50	
Sum Σ	-25.34	+69.80	-69.80	+69.80	-69.80	+25.34
Conventional moments	-25.34	-69.80	-69.80	-69.80	-69.80	-25.34

Moment distribution process (explanation):

- The frame is stretched out and equivalent continuous beam is drawn.
- The FEM & DF are calculated and entered in the moment distribution table.
- The distribution & C.O process is continued up to IV distribution.
- Since the C.O moments after III distribution is very small, the process is stopped at IV distribution.
- The final moments are arrived by algebraic sum of the in each column.
- To get back the conventional moments, by changing the sign of moments at the left end of each span.

BMD:

Free BMD (Sagging):

Max free BM for span AB = $\frac{+Wl}{4} = \frac{+40\times8}{4}$

$= +80\text{kNm}$

Max free BM for span BC = $\frac{+wl^2}{8} = \frac{+30\times6^2}{8}$

$= +135\text{kNm}$

Max free BM for span CD = $\frac{+Wl}{4} = \frac{+40\times8}{4}$

$= +80\text{kNm}$

Fixed BMD (Hogging):

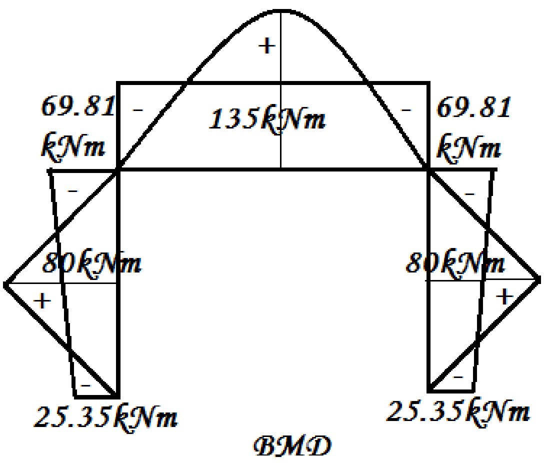
M_A = -25.34kNm.

M_B = -69.80kNm.

M_C = -69.80kNm.

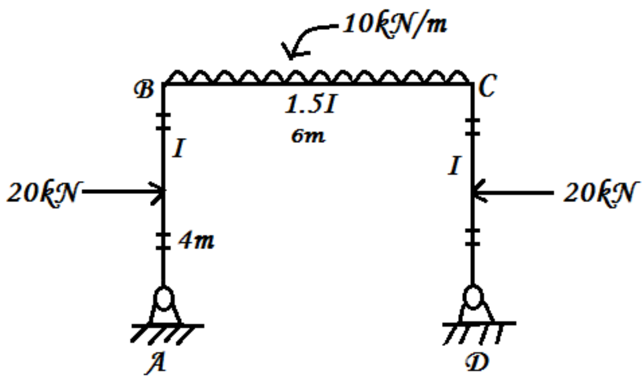
M_D = -25.34kNm.

Final BMD:



WORKED EXAMPLE 2

1) Analyse the portal frame shown in fig. by moment distribution method and draw BMD.

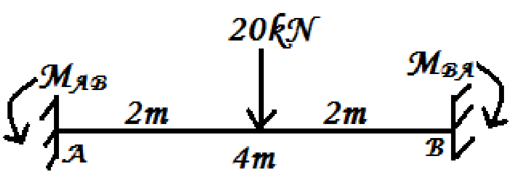


EI not constant: **I** varying.

The frame is stretched & equivalent continuous beam is drawn.

Step 1: FEM

Span AB



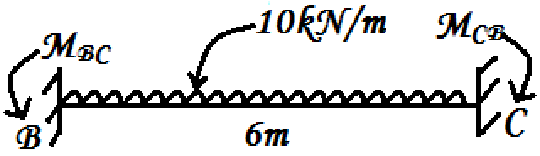
$M_{AB} = \frac{-Wl}{8} = \frac{-20 \times 4}{8}$

$= -10kNm$

$M_{BA} = \frac{+Wl}{8} = \frac{+20 \times 4}{8}$

$= +10kNm$

Span BC



$M_{BC} = \frac{-wl^2}{12} = \frac{-10 \times 6^2}{12}$

$= -30kNm$

$M_{CB} = \frac{+wl^2}{12} = \frac{+10 \times 6^2}{12}$

$= +30kNm$

Span CD (symmetrical)

$M_{CD} = -10kNm$

$M_{DC} = +10kNm$

Step 2: Distribution factors:

1) Joints (B, C)

Joint	Member	Stiffness	Relative stiffness (k)	Sum Σk	Distribution factor(D.F) $(\frac{K}{\sum K})$
B	BA BC	$K_{BA} = \frac{3EI}{l}$ $= \frac{3EI}{4}$ (A is hinged) $K_{BC} = \frac{4EI}{l}$ $= \frac{4E(1.5I)}{l}$ $= \frac{6EI}{6}$ (c is fixed)	$K_{BA} : K_{BC}$ $\frac{3EI}{4} : \frac{6EI}{6}$ LCM is 12 units 9: 12 i.e. 3:4	$=K_{BA} + K_{BC}$ $= 3 + 4$ $= 7$	$D.F_{BA} = \frac{K_{BA}}{\Sigma k}$ $= 3/7$ $D.F_{BC} = \frac{K_{BC}}{\Sigma k}$ $= 4/7$

C	CB	$K_{CB} = \frac{4EI}{l}$ $= \frac{4E(1.5I)}{6}$ $= \frac{6EI}{6}$ (B is fixed)	$K_{CB} : K_{CD}$ $\frac{6EI}{6} : \frac{3EI}{4}$ 12:9 4:3	4 + 3 = 7	$D.F_{CB} = 4/7$
	CD	$K_{CD} = \frac{3EI}{l}$ $= \frac{3EI}{4}$ (D is hinged)			$D.F_{CD} = 3/7$

(2)End supports (A, D)

A & D are hinged. ∴ $DF_{AB} = 1$ $DF_{DC} = 1$

Step 3: Moment distribution table:

A B C D

Member	AB	BA	BC	CB	CD	DC
Distribution factors	1	3/7	4/7	4/7	3/7	1
FEM	-10.00	+10.00	-30.00	+30.00	-10.00	+10.00
Release A & D & C.O to B, C	+10.00	+5.00			-5.00	-10.00
Initial/ Adjusted FEM	0.00	+15.00	-30.00	+30.00	-15.00	0.00
I distribution		+6.43	+8.57	-8.57	-6.43	
C.O (from B to C & C to B)			-4.29	+4.29		
II distribution		+1.84	+2.45	-2.45	-1.84	
C.O			-1.23	+1.23		
III distribution		+0.53	+0.70	-0.70	-0.53	
C.O			-0.35	+0.35		
IV distribution		+0.15	+0.20	-0.20	-0.15	
Final moments(Σ)	0.00	+23.95	-23.95	+23.95	-23.95	0.00
Conventional moment	0.00	-23.95	-23.95	-23.95	-23.95	0.00

Moment distribution process (explanation):

- The given frame is stretched and an equivalent continuous beam is drawn.
- The FEM & DF are calculated and entered in the moment distribution table.

- Since A & D are hinged , A & D are released (balanced) to make the moments zero, and half the moment is C.O to B from A and IIly from D to C. Thus the FEM's are initially adjusted. Algebraic sum at the end (Σ) should start from here only.
- Distribution and C.O process are done at B & C up to IV distribution.
- Since the C.O moments are gradually small, the process is stopped at IV distribution.
- For final moment, algebraic sum of moments from adjusted FEM are done.
- To get back the conventional moments the signs are changed at the left of each support.

BMD:

Free BMD (Sagging):

Max free BM for span AB = $\frac{+Wl}{4} = \frac{+20 \times 4}{4}$

$= +20\text{kNm}$

Max free BM for span BC = $\frac{+wl^2}{8} = \frac{+10 \times 6^2}{8}$

$= +45\text{kNm}$

Max free BM for span CD = $\frac{+Wl}{4} = \frac{+20 \times 4}{4}$

$= +20\text{kNm}$

Fixed BMD (Hogging):

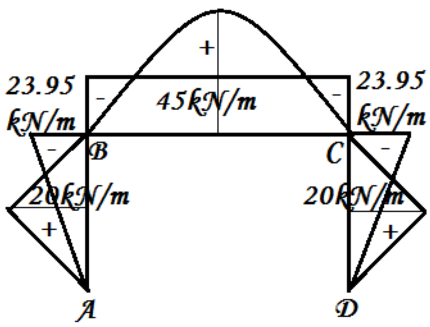
$M_A = 0$ (hinged).

$M_B = -23.95$kNm.

$M_C = -23.95$kNm.

$M_D = 0$ (hinged).

Final BMD:

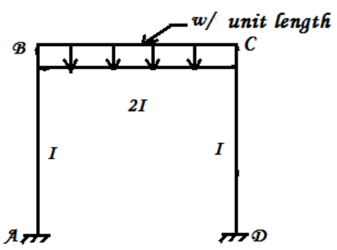
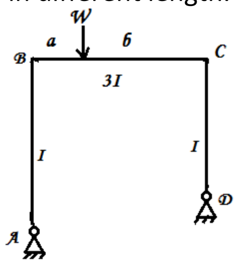


HIGH LIGHTS

1. **Portal Frame:**

A Frame consisting of beam resting on columns with rigid joints are known as portal frame.

2. Symmetrical and Unsymmetrical portal frames:

Sl.no	Symmetrical Portal Frame	Unsymmetrical Portal Frame
1.	Both the columns are same length. 	Columns are in different length. 
2.	Modulus of elasticity is same.	Modulus of elasticity may be different.
3.	Moment of inertia is same for both the columns.	Moment of inertia may be different.
4.	Both the end conditions are same.	End conditions are different.
5.	Loading is symmetrical.	Loading is unsymmetrical.

3. Sway type Portal Frame:

In case of unsymmetrical portal frame, the frame deflects horizontally. The frame having horizontal deflection is called sway type portal frame. In this case sway moments are considered. If the symmetrical portal frame is loaded asymmetrically sway moments are also considered.

These may be classified into

i) Pure sway frame

ii) General sway frame

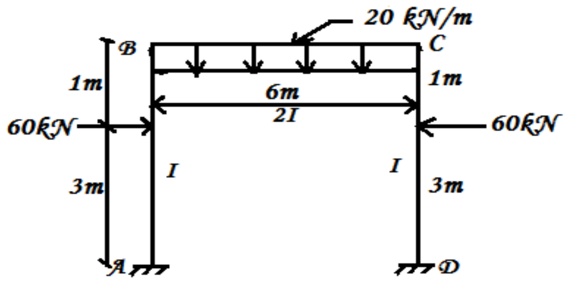
4. Non – sway type portal frame:

In case of symmetrical portal frame with symmetrical load horizontal deflection will not occur, this frame is called non – sway type portal frame.

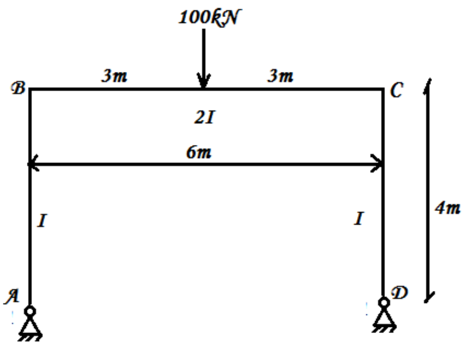
QUESTIONS

Ten mark Questions

1. Analyse the portal frame shown in figure by moment distribution method and draw the BMD.



2. Analyse the portal frame shown in figure and draw the bending moment diagram by moment distribution method.



4. COLUMNS AND STRUTS

4.1 COLUMNS AND STRUTS

columns and struts – Definition – Short and Long columns – End conditions – Equivalent length / Effective length - Slenderness ratio – Axially loaded short column – Axially loaded long column – Euler's Theory of long columns – Derivation of expression for critical load of columns with hinged ends-Expressions for other standard cases of end conditions (separate derivations not required)-Problems- Derivation of Rankine's formula for crippling load of columns-Factor of safety-safe load on columns-simple problems

Chapter 4

4.1. STRUTS AND COLUMN

Any member of structure subjected to axial compressive force is known as strut. A strut inclined at 90^{o} to the horizontal (i.e. a vertical strut) is known as column, pillar or stanchion.

4.1.1.DEFINITIONS

(a) **Column:** A long vertical slender member or bar subjected to an axial compressive force is known as column.

(b) **Strut**: A slender member in any position other than vertical subjected to axial compressive force is known as strut.

(c) **Slenderness ratio:** It is the ratio of length of column to the least radius of gyration. Slenderness ratio has no unit.

Slenderness ratio =Length of column/Least radius of gyration=L/K

It represents the extent to which the column is long and slender .As the slenderness ratio of a column increases its compressive strength decreases. A slenderness ratio of 200 is extremely large for a column.

(a) Buckling load: The maximum axial compressive load at which the column starts buckling is known as buckling load or crippling load or critical load. Buckling always takes place about the axis having least moment of inertia. The value of buckling load is less than the crushing load.

(b) Safe load: It is the load which a column can withstand safely without any buckling/failure. It is the ratio of buckling load and factor of safety.

Safe load=Buckling load/Factor of safety

(c) Buckling factor: Buckling factor is the ratio of equivalent length to the least radius of gyration.

4.1.2. CLASSIFICATION OF COLUMNS

Depending upon the length to diameter ratio or slenderness ratio, a column can be calssified as:

(a) **Short column:** If the lenth of a column is less than 8 times its least lateral dimension then the column is said to be a short column.If the slenderness ratio of a column is less than 32, then the column is also called a shortcolumn.In short column, buckling is negligible and the column fails due to direct crushing only.

(b) **Medium size column:** A column is said to be medium size column if its length varying from 8 times to 30 times of its least lateral dimension. If the slenderness ratio of a column lies between 32 to 120, then the column is also said to be medium size or intermediate column. In these column, both buckling as well as direct stresses are of significant value. In the design of medium size column, both the stresses are taken into account.

(c) **Long column:** A column is said to be long column if its length is more than 30 times the least lateral dimension or the slenderness ratio of the column is more than 120. In this type of column, direct compressive stress is very small as compared to buckling stress, i.e., the failure is only due to buckling stress. Hence, long columns are designed to withstand buckling stresses.

4.1.3. FAILURE OF COLUMN

The failure of a column takes place due to any of the following stresses:

i. Direct compressive stress
ii. Buckling stress
iii. Combination of direct and buckling stresses

(A)Failure of short column

Consider a short column of cross-sectional area 'A' subjected axial compressive load p, then

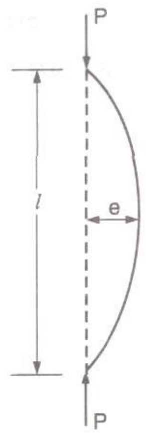
$$\text{Compressive stress} = \frac{P}{A}$$

If the compressive load is gradually increased, a stage will be reached at which the column will be at a point of failure by crushing. The stress inducted in the column corresponding to this load is called crushing stress and the load is known as crushing load (p_c).

$$\text{Crushing stress} = \frac{\text{Crushing load}}{\text{cross section area}} = \frac{P}{A}$$

(B) Failure of a long column:

Figure 4.1 shows a long column of cross-sectional area 'A' subjected to an axial compressive load p.In case of long column, the failure is due to crushing as well as buckling (bending). The load at which the column just buckles is known as buckling load. The buckling load is less than the crushing load.



Let, p = Axial compressive load
l = length of the column
A = Cross-sectional area
e = Maximum bending at the centre of the column
z = Section modulus in the axis of bending

Direct stress, $\sigma_d = \frac{P}{A}$

Bending stress, $\sigma_b = \frac{P.e}{z}$

Stresses at the mid-section of the column are:

Maximum stress, $_{max} = \boldsymbol{\sigma}_b + \boldsymbol{\sigma}_d$

Maximum stress, $\boldsymbol{\sigma}_{min} = \boldsymbol{\sigma}_{b} - \boldsymbol{\sigma}_{d}$
If the maximum stress is more than crushing stress, the column will fail. In long columns, direct compressive stress is negligible as compared to buckling stress. Hence, very long columns are subjected to buckling stresses only.

4.1.4. END CONDITIONS OF COLUMN

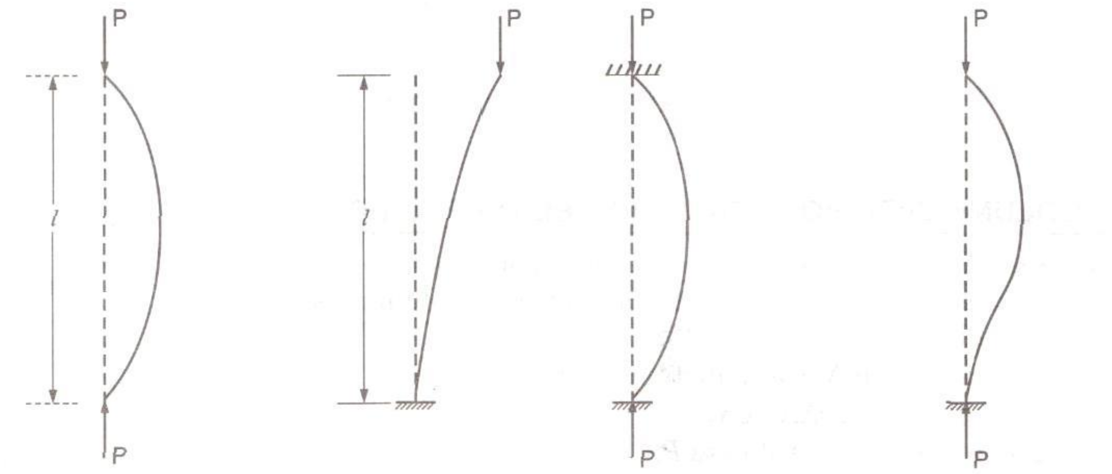
The following end conditions of columns are important:

(a) Both the ends hinged [Fig. 4.2(i)]

(b) Both the ends fixed [Fig. 4.2(ii)]

(c) One end fixed and other end hinged [Fig.4.2(iii)]

(d) One end fixed and other free [Fig.4.2(iv)]



(i) Both ends hinged (ii) One end fixed and other free (iii) Both end fixed (iv) One end fixed and other hinged

Fig.4.2

4.1.5 EQUIVALENT LENGTH OR EFFECTIVE LENGTH

Equivalent length is the length of the long column which is actually involved in bending. Equivalent length of a column is also defined as the distance between adjacent point of inflexion*.The equivalent length of a column is obtained by multiplying it with some constant factor 'C'. The constant factor 'C' depends on end conditions of the column. If ι is the actual length of a column, then its equivalent length, L = c x ι
Hence, in case of column with:

(a) Both ends fixed, equivalent length, L = $\frac{\iota}{2}$

(b) Both ends hinged, L =ι

(c) One end fixed and other end free, L = $\frac{2\iota}{\sqrt{2}}$

(d) One end fixed and other hinged, L = $\frac{\iota}{\sqrt{2}}$

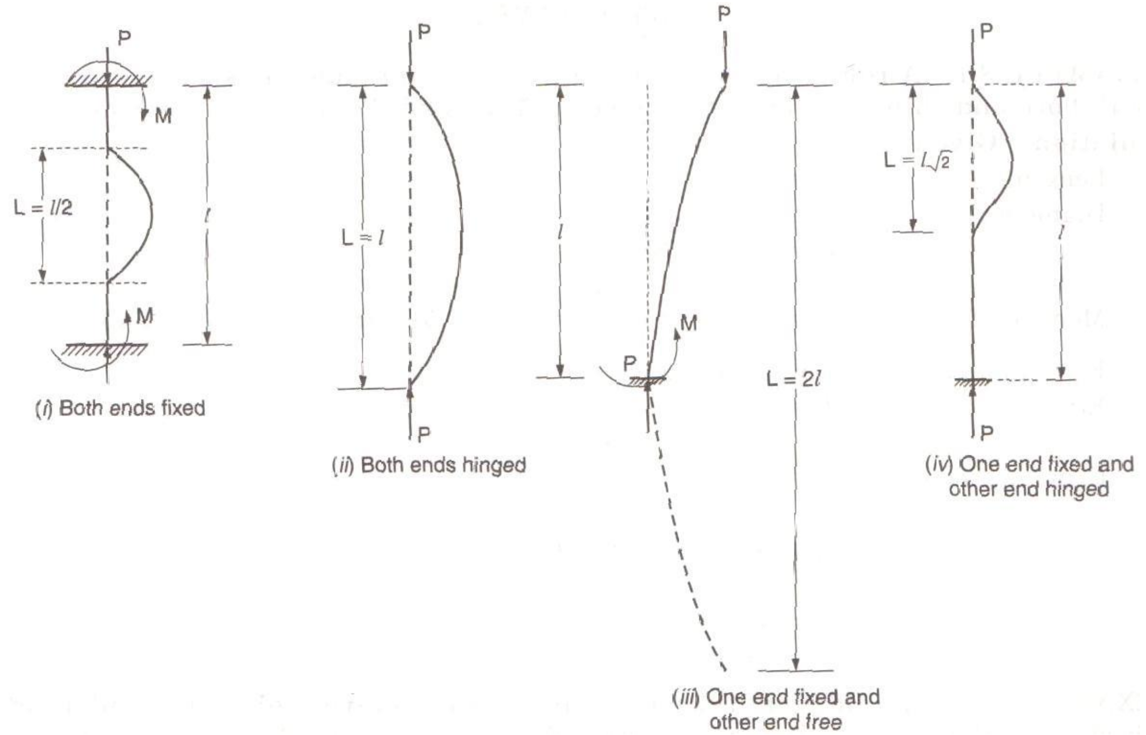


Fig.4.3

Table 4.1. Equivalent Length

S.NO.	**End conditions of column**	**Relation between effective length (L) and actual length (ι)**	**Value of factor 'C'**	**Crippling load in terms of**	
				Actual length	**Effective length**
1.	Both ends fixed	L $= \frac{\iota}{2}$	$\frac{1}{2}$	P=$\frac{4\pi^2 EI}{\iota^2}$	P=$\frac{\pi^2 EI}{L^2}$
2.	Both ends hinged	L = ι	1	P=$\frac{\pi^2 EI}{\iota^2}$	P= $\frac{\pi^2 EI}{L^2}$
3.	One end fixed and other end free	L = 2ι	2	P=$\frac{\pi^2 EI}{4\iota^2}$	P=$\frac{\pi^2 EI}{L^2}$
4.	One end fixed and other hinged	L = $\frac{\iota}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	P=$\frac{2\pi^2 EI}{l^2}$	P=$\frac{\pi^2 EI}{L^2}$

4.2.0. SLENDERNESS RATIO

We have already discussed in art 22.11 that the euler's formula for the crippling load

$$P_E = \frac{\pi^2 EI}{Le^2}$$

We know that the buckling of a column under the crippling load will take place about the axis of least resistance. Now substituting I = Ak^2 (where A is the area and K is the least radius of gyration of the section) in the above equation,

$$p_E = \frac{\pi^2 E(Ak^2)}{L_e^2} = \frac{\pi^2 EA}{\left(\frac{Le}{k}\right)^2}$$

where $\frac{L_e}{k}$ is known as slenderness ratio. Thus slenderness ratio is defined as ratio of equivalent (or unsupported) length of column to the least radius of gyration of the section. slenderness ratio does not have any units.

Note : It may be noted that the formula for crippling load, in the previous articles, have been derived on the assumption the slenderness ratio $\frac{L_e}{k}$ is so large, that the failure of the column occurs only due to bending, the effect of direct stress (i.e., $\frac{P}{A}$) being negligible.

4.2.1.AXIALLY LOADED SHORT COLUMN

If the line of action of load coincides with the axis of the column, the column is called an **axially loaded column.** The load passes through the Centroid of the column section. They are also known as centrally loaded columns or concentrically loaded columns.

Consider a short column subjected to axial compression P.

Compressive stress, σ_c = Load/Area = P/A

If the compression is increased, the column fails by crushing. The load corresponding to this crushing is called **crushing load.** All short columns fail by crushing.

Crushing load, $P_c = \sigma_c * A$

Where σ_c= Ultimate crushing stress in N/mm^2

A = Area of cross section of the column in mm^2

4.2.2. AXIALLY LOADES LONG COLUMN

Consider a long slender column, perfectly straight, subject to axial compression (P) as shown in fig 4.1.1. For small values of P, the column remains straight. When the axial load P is gradually increased the column starts deflecting laterally (Buckle). The column will be under stable equilibrium upto a particular stage and lateral deflection disappear on the removal of load (P). Further increase in load beyond this stage affect the stability of the column and lead to failure by lateral buckling.

The axial load just sufficient to keep the column in stable equilibrium with slight deflected shape is called **BUCKLING LOAD or CRIPPLING LOAD or CRITICAL LOAD.** The lateral deflection of the column is known as **BUCKLING** or (lateral bending)

At buckling load the stress in the column material will be well with in the proportional limit. Depending upon the flexural rigidity (EI), the column will buckle about a plane of least moment of inertia of the section in a direction perpendicular to the axis. Hence stability is more important than strength in the design of columns.

4.2.3. COMPARISONS BETWEEN AXIALLY LOADED SHORT COLUMN AND LONG COLUMN

S.No.	Axially loaded short column	Axially loaded long column
1	Short column has slenderness ratio less than 12	Long column has slenderness ratio greater than 12
2	Failure is due to crushing	Failure is due to buckling
3	The cross section of short column is more	The cross section of long column is less compared to short column

4.2.4. FORMULAE FOR FINDING BUCKLING LOAD IN COLUMNS

The following formulae are used to find out buckling load in columns:

(a) Euler's formula
(b) Rankine Gordon formula
(c) Johnson's parabolic formula
(d) IS formula
(e) Straight line formula.

4.2.5. ASSUMPTIONS MADE IN THE EULER'S THERY

Euler's formula for crippling load is based on the following assumptions:

(e) The section of the column is uniform throughout its length.
(f) The column is initially perfectly straight and axially loaded.
(g) The column material is homogeneous and isotropic.
(h) The column material is perfectly elastic and obeys Hooke's law.
(i) The length of the column is very large as compared to the lateral dimensions.
(j) The self-weight of the column is neglected.
(k) The direct stress is very small as compared to bending stress.
(l) The column will fail by buckling alone.

Sign Convention for Bending Moment: A bending moment is taken as positive if it bends the column with its convexity towards the actual centre line as shown in figure 4.3(i).

A bending moment is taken as negative if it bends the column with its concavity towards the actual centre line, as shown in Fig. 4.3(ii).

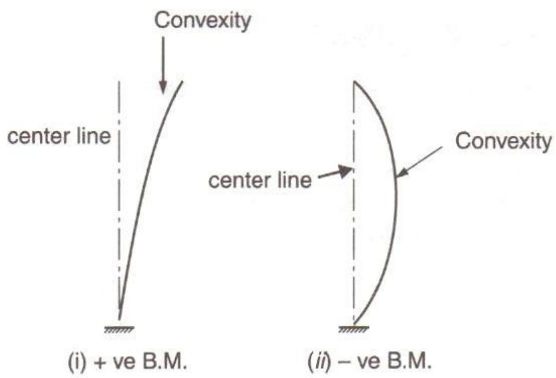


Fig.4.4

4.2.6. COLUMN WITH BOTH ENDS HINGED OR PINNED

Consider a column AB of length 'l' hinged at both its ends A and B ana carries an axial crippling load i.e., load at which the column just buckles, as shown in fig. 4.4.

Consider a section X-X at a distance x from B.
Let the deflection at XX be y.
Bending moment at XX due to p,

M = -Py
[BM is –ve as per the sirn convention]
We know,

$EI\frac{d^2 y}{dx^2} = M$ = -py

EI $\frac{d^2 y}{dx^2}$ = -py

$\frac{d^2 y}{dx^2} = -\frac{py}{EI} = -K^2$y

Where

$K^2 = \frac{P}{EI}A$

$\frac{d^2 y}{dx^2} + K^2 y = 0$

Solution of this differential equation is given by

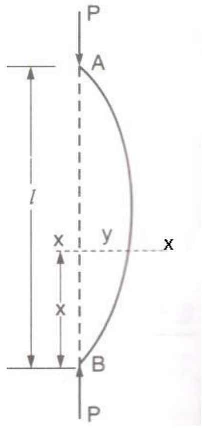


Fig.4.5

y = A cosKx+B sin Kx

where
A and B are constants
or

$$y = A\, cos\left(\sqrt{\frac{P}{EI}}\right) + B \sin\left(x\sqrt{\frac{P}{EI}}\right)$$

At B,

y = 0, x = 0Type equation here.

∴ A = 0$A = \pi r^2$

At A,

y =0, x = l

∴ 0 = B sin l $\sqrt{\frac{P}{EI}} A = \pi r^2$

Or

$$\sin\left(\iota\sqrt{\frac{P}{EI}}\right) = 0$$

∴ $\iota\sqrt{\frac{P}{EI}}$ = 0, π, 2π, 3π…

Considering the least significant value, we get

$$\iota\sqrt{\frac{P}{EI}} = \pi$$

Or

$$\sqrt{\frac{P}{EI}} = \frac{\pi}{\iota}$$

Squaring both sides we get,

$$\frac{P}{EI} = \frac{\pi^2}{\iota^2}$$

$$P = \frac{\pi^2 EI}{\iota^2}.$$

4.2.7. LIMITATION OF EULER'S FORMULA

we have discussed in Art. 32.12 that the Euler's formula for the crippling load,

$$p_E = \frac{\pi^2 EA}{\left(\frac{L_e}{k}\right)^2}$$

Euler's crippling stress,

$$\sigma_E = \frac{P}{A} = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2}$$

A little consideration will show that the crippling stress will be high, when the slenderness ratio is small. We know that the crippling stress for a column cannot be more than the crushing stress of the column material. It is thus obvious that the Euler's formula will give the value of crippling stress of the column (equal to the crushing stress of the column material) corresponding to the slenderness ratio. Now consider a mild steel column. We know that the crushing stress for the mild steel is 320 MPa or 320 N/m^2 and young's modulus for the mild steel is 200 GPa or 200x10^3 N/mm^2.

Now equating the crippling stress to the crushing stress,

$$320 = \frac{\pi^2 E}{\left(\frac{L_e}{k}\right)^2} = \frac{\pi^2 \times (200 \times 10^3)}{\left(\frac{L_e}{k}\right)^2}$$

$$\left(\frac{L_e}{k}\right)^2 = \frac{\pi^2 \times (200 \times 10^3)}{320}$$

$$\frac{L_e}{k} = 78.5 \text{ say } 80$$

Thus if the slenderness ratio is less than 80 the Euler's formula for a mild steel column is not valid.

Sometimes, the column , whose slenderness ratio is more than 80 are known as long columns and those whose slenderness ratio is less than 80 are known as short column. It is thus obvious that the Euler's formula holds good only for long columns.

Note: In the Euler's formula for crippling load , we have not taken into account the direct stresses induced in the material due to the load , (which increases gradually from zero to its crippling load). As a matter of fact, the combined stress, due to direct load and slight bending reaches its allowable value at a load, lower than that require for bulking; and therefore this will be the limiting value of the safe load.

EXAMPLE 4.1. A mild steel tube 4 m long, 30 mm internal diameter and 4 mm thick is used as a strut. Determine the safe compressive loads when this strut is used with the following end conditions :

(i) Both ends are hinged

(ii) Both ends are fixed

Take the factor of safety = 3 and E = 2 x 10^5 N/mm.

Solution:

Given:

Actual length, ι = 4 m = 4000 mm

Internal diameter, d_i = 30 mm

Thickness, t = 4 mm

∴ Outer diameter, d_0 = Internal diameter + 2t

= 30 + 2 x 4 = 38 mm

Moment of inertia,

$$I = \frac{\pi}{64}(d_0^4 - d_i^4)$$

$$= \frac{\pi}{64}(38^4 - 30^4)$$

$$= \frac{\pi}{64}(2085136 - 810000)$$

$$= 62593.09 \text{ mm}^4$$

Let the crippling load be P.

(i) When both ends are hinged

Effective length,

$$L = \iota = 4000 \text{ m}$$

Crippling load,

$$P = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2 \ X\ 2X\ 10^5\ X\ 62593.09}{(4000)^2} = 7722.11 \text{ N}$$

$$\text{Safe load} = \frac{Crippling\ load}{Factor\ of\ safety} = \frac{7722.11}{3}$$

= 2574.04 N Ans.

(ii) When both ends are fixed

Effective length,

$$L = \frac{\iota}{2} = \frac{4000}{2} = 2000 \text{ mm}$$

Crippling load,

$$P = \frac{\pi^2 EI}{L^2},$$

$$P = \frac{\pi^2\ X\ 2\ X\ 10^5\ X\ 62593.09}{(2000)^2} = \frac{1.235\ X 10^{11}}{4\ X\ 10^6}$$

= 30875 N

Safe load = $\frac{Crippling\ load}{FOS}$ =$\frac{30875}{3}$

=10291.67 N **Ans.**

EXAMPLE 4.2. Find the maximum length of a solid mild steel rod having diameter 40 mm used as a column with both ends fixed to carry a load of 20 KN. Allow factor of safety = 3. Take E = 2 x 105 N/mm2.

Solution:

Given :

Diameter, d = 40 mm
Safe load = 20 KN
FOS = 3
E = 2 X10^5 N/mm

Moment of inertia,

I = $\frac{\pi}{64}d^4$ = $\frac{\pi}{64}(40)^4$

= 125663.71 mm^4

Crippling load,

P = Safe load x FOS

= 20 X 3 = 60 KN = 60000 N

Let the effective length be L and actual length be l

P = $\frac{\pi^2\ EI}{L^2}$

60000 = $\frac{\pi^2\ X\ 2\ X10^5\ X\ 125663.71}{L^2}$

L^2 = $\frac{\pi^2\ X\ 2\ X10^5\ X\ 125663.71}{60000}$

L^2 = 4134170.35

∴ L =$\sqrt{4134170.2}$ = 2033.27 mm

For a column with both end fixed

L = $\frac{\iota}{2}$ or $\iota =$ Actual length = 2L

= 2 X 2033.26 = 4066.54 mm

∴ ι = 4.1 m **Ans.**

EXAMPLE 4.3. A solid round bar 4 m long and 50 mm in diameter was found to extend 4.6 mm under a tensile load of 50 KN. This bar is used as a strut with both ends hinged. Determine the buckling load for the bar and also the safe load taking factor of safety as 4.

Solution.

Given:

Actual length of bar, ι $=$ 4 m = 4000 mm

Change in length, $d\iota$ = 4.6 mm
Tensile load, P = 50 KN = 50000 N
FOS = 4
Diameter of bar, d = 50 mm

Area of cross-section of bar,

$A = \frac{\pi}{4} X (50)^2 = 1963.5 \text{ mm}^2$

Stress = $\frac{P}{A} = \frac{50000}{1963.5} = 25.46 \text{ N/mm}^2$

Strain = $\frac{dl}{l} = \frac{4.6}{4000} = 0.00115$

Modulus of elasticity,

$E = \frac{Stress}{Strain} = \frac{25.46}{0.00115} = 2.2 \times 10^4 \text{ N/mm}^2$

Moment of inertia,

$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} (50)^4$

$= 306796.16 \text{ mm}^4$

Effective length,

$L = \iota = 4000$ mm

Let P_c be the buckling load

We know,

$P_c = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 X 2.2 X 10^4 X 306796.16}{(4000)^2}$

$= \frac{6.66 X 10^{10}}{16 X 10^6} = 4162.5$ N Ans.

$= \frac{Buckling\ load}{FOS}$

$= \frac{4.162.5}{4}$ 1040.63 N **Ans.**

EXAMPLE 4.4. Find the safe load for an elastic column made of a solid steel rod of dia 20 mm and length 1.5 mm. It is fixed at both ends. The factor of safety is 2.5 and modulus of elasticity for the rod is 210 GPa.

Solution.

Given:

Diameter, d = 20 mm
Length, ι = 1.5 mm = 1500 mm
FOS = 2.5

$E = 210 \text{ GPa} = 210 \times 10^9 \text{ Pa} = 210 \times 10^9 \text{ N/m}^2$

$= \frac{210 X 10^9}{10^6} = 210 X 10^3 \text{ N/mm}^2$

Moment of inertia,

$I = \frac{\pi d^4}{64} = \frac{\pi(20)^4}{64}$ = 7853.98 mm^4

Column is fixed at both the ends

∴ Effective length,

$L = \frac{\iota}{2} = \frac{1500}{2}$ = 750 mm

Let P be the crippling load,

We know by Euler's formula

Crippling load

$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 X 210 X 10^3 X 7853.98}{(750)^2}$= 28939.19 N

Safe load $= \frac{Crippling\ load}{FOS} = \frac{28939.19}{2.5}$

= 11575.68 N **Ans.**

EXAMPLE 4.5. A column of timber saction 150 mm x 200 mm is 6 m long both ena being fixed. Find the safe load for the column. Use Euler's formula and allow a factor of safety of 3. Take E = 17500 N/ mm2.

Solution:

Given:

Width, b = 150 mm
Depth, d = 200 m
Length, ι = 6 m = 6000 mm
FOS = 3
E = 17500 N/mm^2

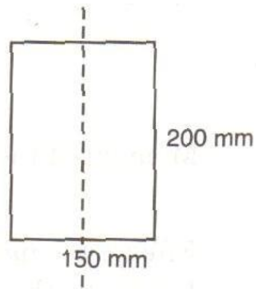


Fig.4.6

Least Moment of inertia,

$I = \frac{db^3}{12} = \frac{200 X 150^3}{12}$

= 5.625 X 10^7 mm^4

Effective length of column when both ends are fixed,

$L = \frac{\iota}{2} = \frac{6000}{2}$ = 3000 mm

Let P be the crippling load.

By Euler's formula

Crippling load,

$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 X 17500 X 5.625 X 10^7}{(3000)^2}$

= 1079488 N

Safe load = $\frac{Crippling\ load}{FOS} = \frac{1079488}{3}$ = 359829.33 N

= 360 KN Ans.

EXAMPLE. 4.6. Compare the ratio of the strength of a solid steel column to that of a hollow of the same cross-sectional area. The internal diameter of the hollow column is ¾ of the external diameter. Both the columns have the same length and are pinned at the ends.

Solution.

Hollow circular column:

Let D and d be the external and internal diameter of hollow column respectively.

$$d = \frac{3}{4} D$$

Moment of inertia,

$$I_h = \frac{\pi}{64}(D^4 - d^4)$$

$$= \frac{\pi}{64}\left\{D^4 - \left(\frac{3}{4}D\right)^4\right\} = \frac{\pi}{64} \times D^4\left(1 - \frac{81}{256}\right)$$

$$= \frac{175\pi\ X\ D^4}{64\ X\ 256}$$

Cross sectional area,

$$A_h = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\left\{D^2 - \left(\frac{3}{4}D\right)^2\right\}$$

$$= \frac{\pi}{4} \times D^2\left(1 - \frac{9}{16}\right) = \frac{\pi}{4} \times D^2 \times \frac{7}{16}$$

$$= \frac{7\pi D^2}{64}$$

Crippling load,

$$P_h = \frac{\pi^2\ EI}{L^2} = \frac{\pi^2 X\ E\ X\ 175\pi D^4}{64\ X\ 256\ L^2}$$

Solid Column:

Let d_1 be the diameter of the solid column

Cross-sectional area,

$$A_s = \frac{\pi}{4} d_1^2$$

According to question $A_s = A_h$

$$\frac{\pi}{4} d_1^2 = \frac{7\pi D^2}{64}$$

$$d_1^2 = \frac{7\pi D^2}{64} \times \frac{4}{\pi} = \frac{7}{16} D^2$$

$$d_1 = \frac{\sqrt{7}}{4} D$$

Moment of inertia,

$$I_s = \frac{\pi}{64} d_1^4 = \frac{\pi}{64} \times \left(\frac{\sqrt{7}}{4}D\right)^4 = \frac{\pi}{64} \times \frac{49}{16} D^4$$

Crippling load,

$$P_s = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 E\ X\ \pi\ X 49 D^4}{L^2 X\ 64\ X\ 16}$$

Dividing (i) and (ii), we get

$$\frac{P_S}{P_h} = \frac{\pi^2 X\, E\, \pi\, X 49 D^4}{L^2 X\, 64\, X\, 16} \text{ x } \frac{64\, X\, 256\, L^2}{\pi^2 X\, E\, X\, 175 \pi D^4} = \frac{49\, X\, 16}{175} = 4.48$$

EXAMPLE 4.7. A steel rod 5m long and of 40 mm diameter is used as a column, with on end fixed and the other free. Determine the crippling load by Euler's formula. Take E as 200 GPa

Solution.

Given:

Length(l) = 5m = 5 x 10^3 mm;

Diameter of column (d) = 40 mm

modulus of elasticity (E0 = 200 GPa = 200 x 10^3 N/mm^2.

We know that moment of inertia of the column section,

$$I = \frac{\pi}{64} \text{ x } (d)^4 = \frac{\pi}{64} \text{ x } (40)^4 = 40{,}000\pi \text{ mm}^4$$

Since the column is fixed at one end and free at the other, therefore equivalent length of the column,

$$L_e = 2\,l = 2 \text{ x } (5 \text{ x } 10^3) = 10 \text{ x } 10^3 \text{ mm}$$

Euler's crippling load,

$$P_E = \frac{\pi^2 EI}{Le^2} = \frac{\pi^2 \text{ x } (200 \text{ x } 10^3) \text{x } (40000\pi)}{(10 \text{x} 10^3)^2} = 2480 \text{ N}$$

= 2.48 kN **Ans.**

EXAMPLE 4.8. A hollow alloy tube 4m long with external and internal diameter of 40mm and 25mm respectively was found to extend 4.8mm under a tensile load of 60KN. Find the bulking load for the tube with both ends pinned. Also find the safe load on the cube, taking a factor of safety as 5

Solution:

Given :

length l	= 4m,
External diameter of column (D)	= 40mm ;
Internal diameter of column (d)	= 25mm;
Deflection (δl)	= 4.8mm;
Tensile load	= 60KN
Factor of safety	= 5

Buckling load for the tube,

We know that area of the tube,

$$A = \frac{\pi}{4} \text{ X } (D^2 - d^2) = \frac{\pi}{4}((40)^2 - (25)^2) = 765.8 \text{ mm}^2$$

And moment of inertia of the tube,

$$I = \frac{\pi}{64} \text{ X } (D^4 - d^4) = \frac{\pi}{64} \text{ X } ((40)^4 - (25)^4) = 106488.95 \text{mm}^4$$

We also know that the strain in the alloy tube,

$$e = \frac{\delta l}{l} = \frac{4.8}{4\times10^3} = 0.0012$$

and modulus of elasticity for the alloy,

$$E = \frac{\text{Load}}{\text{Area} \times \text{Strain}} = \frac{60\times10^3}{765.8\times0.0012} = 65291.20 \text{ N/mm}^2$$

since the column is pinned at its both ends, therefore equivalent length of the column,

$$L_e = l = 4\times10^3$$

Euler's bukling load,

$$P_E = \frac{\pi^2 EI}{Le^2} = \frac{\pi^2\times65291.20\times106488.95}{(4\times10^3)^2} = 4288.83 \text{ N}$$

$= 4.29$ KN Ans.

Safe load for the tube

We also know that safe load for the tube

$$= \frac{\text{Buckling load}}{\text{Factor of safety}} = \frac{4.29}{5} = 0.858 \text{ KN} \quad \text{Ans.}$$

EXAMPLE 4.9. Compare the ratio of the strength of a solid steel column to that of a hollow of the same cross-sectional area. The internal diameter of the hollow column is¾ of the external diameter. Both the columns have the same length and are pinned at both ends.

Solution.

Area of solid steel column $A_s = A_H$ (where A_H = Area of hollow column);
internal diameter of hollow column (d) = 3 D/4 (where D = External diameter)
length of solid column (I_S) = I_H = (where I_H = Length of hollow column).

Let D_1 = Diameter of the solid column,
k_H = Radius of gyration for hollow column and
k_S = Radius of gyration for solid column.
Since both the columns are pinned at there both ends, therefore equivalent length of the solid column,

$$L_S = I_S = L_H = I_H = L$$

We know that Euler's crippling load for the solid column,

$$P_s = \frac{\pi^2 EI}{{L_H}^2} = \frac{\pi^2 E.A_S.{k_S}^2}{L^2} D_1$$

Dividing equation (ii) by (i),

$$\frac{P_H}{P_S} = \left(\frac{k_H}{k_S}\right)^2 = \frac{\frac{D^2+d^2}{16}}{\frac{D^2}{16}} = \frac{D^2+d^2}{{D_1}^2} = \frac{D^2 = \left(\frac{3D}{4}\right)^2}{{D_1}^2}$$

$$= \frac{25\, D^2}{16\, {D_1}^2}$$

Since the cross-sectional areas of the both the columns is equal, therefore

$\frac{\pi}{4}$x D $=\frac{\pi}{4}(D^2 - d^2)$ $=\frac{\pi}{4}\left[D^2 - \left(\frac{3D}{4}\right)^2\right] = \frac{\pi}{4}$ x $\frac{7\,D^2}{16}$

${D_1}^2 = \frac{7\,D^2}{16}$

Now substituting the value of ${D_1}^2$ in equation (iii),

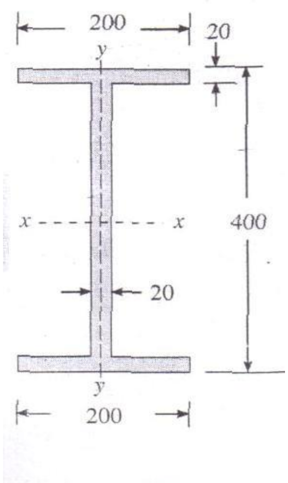
$\frac{P_H}{P_S} = \frac{25\,D^2}{16 \text{ x } \frac{7\,D^2}{16}} = \frac{25}{7}$ **Ans.**

EXAMPLE 4.10.An I section joist 400mm x 200mm x 20mm and 6m long is used as a strut with both ends fixed. What is Euler's crippling load for the column? Take young's modulus for the joist as 200 GPa.

Solution.

Given :

Outer depth (D) = 400 mm;
Outer width (B) = 200 mm;
Length (l) = 6m = 6x10^3mm
modulus of elasticity (E) = 200 GPa = 200 x 10^3N/mm^2.



From the geometry of the figure, we find that inner depth,

d = 400 – (2 x 20) = 360 mm

And inner width,

b = 200 – 20 = 180 mm

We know that moment of inertia of the joist section about X-X axis,

$I_{xx} = \frac{1}{12}[BD^3 - ba^3]$

$= \frac{1}{12}[200 \text{ x } (400)^3 - 180 \text{ x } (360)^3]$mm^4

= 366.8 x 10^6 mm^4

Similarly,

$I_{yy} = \left[2 \text{ x} \frac{2 \text{ x}(200)^3}{12}\right] + \frac{360 \text{ x } (20)^3}{12}$ mm^4

= 2.91 x 10^6 mm^4

Since I_{yy} is less than I_{xx}, therefore the joist will tent to buckle in Y-Y direction. Thus, we shall take the value of I_{yy} = 2.91 x 10^6 mm^4. Moreover, as the column is fixed at its both ends, therefore equivalent length of the column.

$L_e = \frac{l}{2} = \frac{(6 \text{ x} 10^3)}{2}$ = 3 x 10^3 mm

Euler's crippling load for the column,

$P_E = \frac{\pi^2 EI}{{L_e}^2} = \frac{\pi^2 \text{x } (200 \text{ x } 10^3) \text{ x } (2.91 \text{ x } 10^6)}{(3 \text{ x } 10^3)^2}$ = 638.2 x 10^3 N

= 638.2 KN **Ans.**

EXAMPLE 4.11. A T-section 150 mm x 120 mm x 20 mm is used as a strut of 4 m long with hinged at its both ends. Calculate the crippling load, If Young's modulus for the material be 200 GPa.

Solution.

Given:

Size of T-section = 150 mm x 120 mm x 20 mm;

Length (l) = 4 m = 4 x 10^3 mm

Young's modulus (E) = 200 GPa

= 200 x 10^3 N/mm.

First of all, let us find the centre of the T-section; Let bottom of the web be the axis of reference.

Web

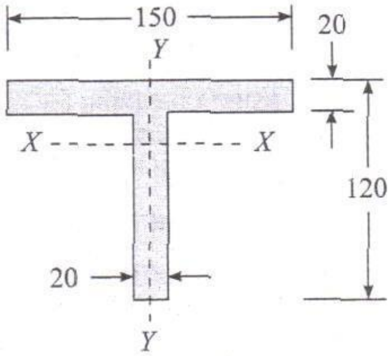
a_1 = 100 x 20 = 2000 mm^2

$y_1 = \frac{100}{2}$ = 50 mm

Flange

a_2 = 150 x 20 = 3000 mm^2

$y_2 = 100 + \left(\frac{20}{2}\right)$ = 110 mm



We know that distance between the centre of gravity of the T-section and bottom of the web

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(2000 \text{ x } 50)+(3000 \text{ x } 110)}{200+3000} = 86 \text{ mm}$$

We also know that moment of inertia of the T-section about X-X axis, **Fig.4.8**

$$I_{xx} = \left(\frac{20 \text{ x } (100)^3}{12} + 2000 \text{ x } (36)^2\right) + \left(\frac{150 \text{ x } (20)^3}{12} + 3000 \text{ x } (24)^2\right) \text{ mm}^4$$

= (4.26 x 10^6) + (1.83 x 10^6) = 6.09 x 10^6 mm^4

Similarly,

$$I_{yy} = \frac{100 \text{ x } (20)^3}{12} + \frac{20 \text{ x } (150)^3}{12} = 5.692 \text{ x } 10^6 \text{ mm}^4$$

Since I_{yy} is less than I_{xx}, therefore the column will tend to buckle in *Y-Y* direction. Thus, we shall take the value of I as I_{yy} = 5.69 x 10^6 mm^4. Moreover, as the column is hinged at its both ends, therefore length of the column,

$L_e = l$ = 4 x 10^3 mm

Euler's crippling load,

$$P_E = \frac{\pi^2 EI}{{L_e}^2} = \frac{\pi^2 \text{x } (200 \text{ x } 10^3) \text{x } (5.69 \text{ x } 10^6)}{(4 \text{ x } 10^3)^2} = 702 \text{ x } 10^3 \text{ N}$$

= 702 kN **Ans.**

4.2.8. RANKINE'S FORMULA

Euler's formula gives correct result only for very long column which fails by buckling. Short column fails by crushing. In practice we come across struts and columns which are neither too short nor long. The failure of the column will be due to the combined effect of crushing and buckling. Rankine devised an empirical formula based on experiments for the collapse load which is applicable for both short and long columns.

Let P = Crippling load by Rankine's formula

Pc = Crushing load or compressive load

P_e= Crippling load by Euler's formula.

Then the Rankine's formula is given by

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_e}$$

Where $P_c = \sigma_C$ x A

= Compressive stress (yield stress)

And

$$P_e = \frac{\pi^2 EI}{L^2}$$

A = cross-sectional area of column

We have,

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_e}$$

$$\frac{1}{P} = \frac{P_e + P_C}{P_C\, P_e}$$

Dividing numerator and denominator of RHS by P_e

$$P = \frac{P_C}{1+\frac{P_C}{P_e}} = \frac{\sigma_C A}{1+\frac{\sigma_C A L^2}{\pi^2\, EI}} \therefore P_c = \sigma_c A \text{ and } P_e = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\sigma_C A}{1+\frac{\sigma_C A L^2}{\pi^2\, E\, X A K^2}} = \frac{\sigma_C A}{1+\frac{\sigma_C}{\pi^2 E}\, X \left(\frac{L}{K}\right)^2} \therefore I = AK^2$$

$$= \frac{\sigma_C A}{1+\alpha\left(\frac{L}{K}\right)^2}$$

where $\frac{L}{K}$ =Slenderness ratio

$\alpha =$ Rankine's constant = $\frac{\sigma_c}{\pi^2 E}$

4.2.9. Applicability

(i) In case of short column, P_e will be large as L is small. As P_e is large, $\frac{1}{P_e}$ will be small enough compared to $\frac{1}{P_C}$. Hence $\frac{1}{P_e}$ may be neglected and equation (i) becomes

$\frac{1}{P} = \frac{1}{P_c}$

Or

P = P_c

Hence, crippling load by Rankine's formulae, for a short column is approximately equal to crushing load.

(ii) In case of long column, P_e will be small as L is large. As P_e is small, $\frac{1}{P_e}$ will be large enough as compared to $\frac{1}{P_c}$. Hence $\frac{1}{P_c}$ may be neglected and equation (i) becomes

$\frac{1}{P} = \frac{1}{P_e}$

$\therefore$ P = P_e

Hence, crippling load by Rankine's formula for a long column is approximately equal to crippling load by Eiler's formula.

So Rankine's formula give satisfactory result for both long and short columns.

Table 4.2.Rankine's constants (α)

S.No	Material	σ_c N/mm^2	$\alpha = \frac{\sigma_c}{\pi^2 E}$
1.	Mild steel	320	$\frac{1}{7500}$
2.	Cast iron	550	$\frac{1}{1600}$
3.	Wrought iron	250	$\frac{1}{9000}$
4.	Timber	50	$\frac{1}{750}$

Table 8.3 Rankine's Critical stress for Mild Steel columns

For σ_c = 320 N/mm^2 and $\sigma = \frac{1}{7500}$

S.No	Slenderness	Rankine's critical N/mm^2
1.	10	316
2.	20	304
3.	30	286
4.	40	264
5.	50	240
6.	60	216
7.	70	193
8.	80	173
9.	90	153
10.	100	137
11.	110	122

12.	120	110
13.	130	98
14.	140	89
15.	150	80
16.	160	72
17.	170	66
18.	180	60
19.	190	55
20.	200	50
21.	210	47
22.	220	43
23.	230	40
24.	240	37
25.	250	34

4.3.0. FACTOR OF SAFETY

Factor of safety is defined as the ratio between crippling load and safe load.

$$\text{Factor of safety } = \frac{\text{Crippling load}}{\text{Safe load}}$$

The values of factor of safety in engineering design varies from 3 and 12.

4.3.1.SAFE LOAD

Factor of safety is defined as the ratio between crippling load and safe load.

$$\text{Safe load } = \frac{\text{Crippling load}}{\text{Factor of safety}}$$

EXAMPLE .4.12. An ISMB 250 Rolled steel joist is to be used as a column 4.0 m long with both ends fixed. Find the safe load on the column allowing a factor of safety of 3. Take σ_c = 320 N/mm2 and α =$\frac{1}{7500}$.

Properties of column section are:

A = 4755 mm^2

I_{xx} = 5.1316 x 10^7 mm^4

I_{yy} = 3.345 x 10^6 mm^4

Solution:

Given :

Actual length of column,

= 4.0 m = 4000 mm

FOS = 3S

σ_c = 320 N/mm^2

$\alpha = \frac{1}{7500}$

A =4755 mm^2

Least moment of inertia,

$I = 3.345 \times 10^6\ mm^4$

Effective length of column,

$L = \frac{l}{2} = \frac{4000}{2} = 2000$ mm

$I = AK^2$

$\therefore\ K = \sqrt{\frac{I}{A}} = \sqrt{\frac{3.345\ X\ 10^6}{4755}} = 26.52$

Rankine's crippling load is given by

$P = \frac{\sigma_C A}{1+\alpha\left(\frac{L}{K}\right)^2} = \frac{320\ X\ 4755}{1+\frac{1}{7500}\ X\left(\frac{2000}{26.52}\right)^2}$

$= \frac{320\ X\ 4755}{1.76} = 864545.45$ N

Safe load $= \frac{\text{Crippling load}}{FOS} = \frac{864545.45}{3}$

$= 288181.82$ N $\approx 288.18\ KN$ Ans.

EXAMPLE.4.13. A hollow cast iron column of external diameter 250 mm and internal diameter 200 mm is 8 m long with one end fixed and the other end hinged. Find the safe load with a factor of safety of 5. Take σc = 550 N/mm2 and α = $\frac{1}{1600}$.

Solution.

Given :

Actual length of column,

= 8 m = 8000 mm

Outer diameter, D = 250 mm

Inner diameter, d = 200 mm

FOS = 5

Compressive stress,

$\sigma_c = 550\ N/mm^2$

Rankine's constant,

$\alpha = \frac{1}{1600}$.

When one end fixed and other end hinged

Effective length of column,

$L = \frac{l}{\sqrt{2}} = \frac{8000}{\sqrt{2}} = 5656.85$ mm

Least moment of inertia,

$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(250^4 - 200^4)$

$= \frac{\pi}{64} \times 2.306 \times 10^9 = 1.132 \times 10^8\ mm^4$

Cross-sectional area,

$A = \frac{\pi}{64}(D^2 - d^2) = \frac{\pi}{64}(250^2 - 200^2)$

$= \frac{\pi}{64} \times 22500 = 17671.46\ mm^2$

$I = AK^2$

K = $\sqrt{\frac{I}{A}}$ = $\sqrt{\frac{1.132\ X\ 10^8}{17671.46}}$ =$\sqrt{6405.81}$ = 80.04

Rankine's crippling load is given by

P= $\frac{\sigma_c A}{1+\alpha\left(\frac{L}{K}\right)^2}$ = $\frac{550X17671.46}{1+\frac{1}{1600}.\left(\frac{5656.85}{80.04}\right)^2}$

= $\frac{9719303}{4.12}$ = 2359054.1 N

Safe load = $\frac{crippling\ load}{FOS}$ = $\frac{2359054.1}{5}$

= 471810.83 N

= 471.81 KN Ans.

EXAMPLE.4.14. Find the Euler's crippling load for a hollow cylindrical steel column of 38mm external diameter and 2.5mm thick. Take length of the column as 2.3m and hinged at its both ends. Take E=205 GPa. Also determine crippling load by Rankine's formula using constants as 335MPa and $\frac{1}{7500}$

Solution.

Given:

External diameter	D = 38mm
Thickness,	= 2.5 mm
Internal diameter	d = 33 mm (38-2x2.5)
Length of the column	ι = 2.3m
Yield stress	σc = 335 MPa =335 N/mm^2
Rankine's Constant	(a) = $\frac{1}{7500}$

Euler's Crippling load

I = $\frac{\pi}{64}(D^4 - d^4)$

I = $\frac{\pi}{64}(38^4 - 33^4)$

= 14.05x10$^3\pi$mm^4

Le = ι = 2.3 x 10^{3}mm

Euler's Crippling load,

P_E= $\frac{\pi^2\ EI}{Le^2}$

P_E= $\frac{\pi^2\ (205\ x\ 10^3\ )\ x\ (14.05\ x10^3\pi)}{(2.3\ x10^3)^2}$ = 16880 N =16.88 kN

Rankine's Crippling load

A = $\frac{\pi}{4}(D^2 - d^2)$

A = $\frac{\pi}{4}(38^2 - 33^2)$

= 88.75πmm^2

k = $\sqrt{\frac{I}{A}}$

k = $\sqrt{\frac{(14.05\ x10^3\pi)}{88.75\pi}}$

= 12.6mm

$P_R = \frac{\sigma_{cs}A}{1+\alpha\left(\frac{Le}{K}\right)^2}$

$P_R = \frac{335x88.75\pi}{1+\frac{1}{7500}\left(\frac{2.3\ x10^3}{K12.6}\right)^2}$

= 17169 N = 17.17 kN

EXAMPLE 4.15. Figure 4.9 shows a built-up column consisting of 150 mm x 100 mm R.S.J. with 120 mm x12 mm plate riveted to each flange.

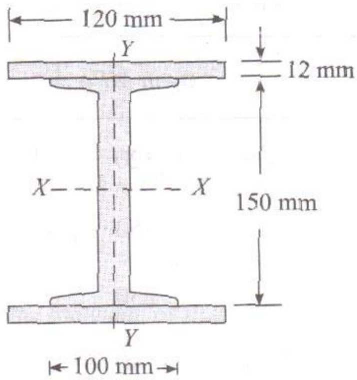


Fig.4.9

Calculate the safe load, the column can carry, if it is 4 m long having one end fixed and the other hinged with a factor of safety 3.5. Take the properties of the joist as Area = 2167 mm2, Ixx = 8.391 x 106 mm4, Iyy= 0.948 x 106mm4 . Assume the yield stress as 315 MPa and Rankine's constant (a) = $\frac{1}{7500}$.

Solution.

Given :

Length of the column (ι)	= 4 m = 4 x 10^3 mm ;
Factor of safety	= 3.5 ;
Yield stress(σ_c)	= 315 MPa = 315 N/mm^2 ;
Area of joist	= 2167 mm^2 ;
Moment of inertia,	
about X-X axis (I_{xx})	= 8.391 x 10^6 ;
about Y-Y axis I_{yy}	= 0.948 x 10^6 mm^4

Rankine's constant (a) $= \frac{1}{7500}$.

From the geometry of the figure, we find that theare of the column section,

A = 2167 + (2 x120 x 12) = 5047 mm^2

And moment of inertia of the column section about X-X axis,

I_{xx}=(8.391 x 10^6) + 2$\left[\frac{120\ X\ (12)^3}{12} + 120\ X 12\ X (81)^2\right]$$mm^4$

= (8.391 x 10^6) + (18.93 x 10^6) = 27.32 x 10^6 mm^4

Similarly,

I_{yy}= (0.948 x 10^6) + 2$\left[\frac{12\ X\ (120)^3}{12}\right]$ mm^4

= (0.948 x 10^6) +(3,456 x 10^6) = 4.404 x10^6 mm^4

Since $I_{yy}$is less than $I_{xx,}$ therefore the column will tend to buckle inY-Y direction.Thus we shall take I equal to I_{yy} = 4.404 x 10^6 mm^4 (i.e., least of two). Moreover as the column is fixed at one end and hinged at the other, therefore equivalent length of the column.

$L_e = \frac{l}{\sqrt{2}} = \frac{4\ X\ 10^3}{\sqrt{2}}$ = 2.83 X 10^3 mm

We know that least radius of gyration,

$K = \sqrt{\frac{I}{A}} = \sqrt{\frac{4.404\ X\ 10^6}{5047}}$ = 29.4 mm

∴ Rankine's crippling load on the column

$P_R = \frac{\sigma_C.A}{1+a\left(\frac{Le}{k}\right)^2} = \frac{315\ X\ 5047}{1+\frac{1}{7500}\left(\frac{2.83\ X\ 10^3}{29.5}\right)^2}$

= 716 x 10^3 N = 716 kN

And safe load on the column

$= \frac{Crippling\ load}{Factor\ of\ safety} = \frac{716}{3.5}$ = 204 kN   Ans.

EXAMPLE 4.16.A column is made up of two channels. ISJC 200 and two 250mm x 10 mm flange plates as show in Fig. 4.10.

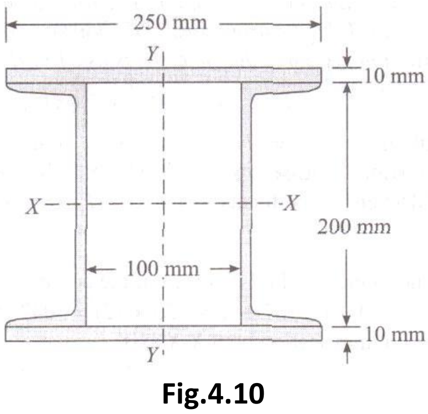


Fig.4.10

Determine by Rankine's formula the safe load, the column of 6 mm length, with both ends fixed, can carry with a factor of safety 4. The properties of one channel are Area = 1777 mm2 ,lxx = 11.612 x 106 mm4 and Iyy = 0.842 x 106 mm4 . Distance of centroid from back to web = 22.7 mm. Take σ_c = 320 MPa and Rankine's constant = $\frac{1}{7500}$

Solution.

Given :

Length of the column (ι) = 6 m = 6 x10^3 mm ;
Factor of safety = 4 ;
Area of channel = 1777 mm^2 ;
Moment of inertia, about X-X axis (I_{xx}) = 11.612 x 10^6 mm^4 ;
Moment of inertia, about Y-Y axis I_{yy}= 0.842 x 10^6 mm^4 ;
Distant of centroid from
the back of web = 22.7 mm ;
crushing stress (σ_c) = 320 MPa = 320 N/mm^2
Rankine's constant (a) = $\frac{1}{7500}$.

From the geometry of the figure, we find that area of the column section,

A = $2[1777 + (250\ X\ 10\ )]$ = 8554 mm^2

And moment of inertia of the column section about X-X Axis.

$$I_{XX} = (2x11.612 \text{ x } 10^6) + 2\left[\frac{250\times10^2}{12} + (250\times10)\times(105)^2\right] mm^4$$

$$= (23.224 \text{ x } 10^6) + (55.167\times10^6) mm^4$$

$$= 78.391\times10^5\ mm^4$$

Similarly, $I_{YY} = 2\left[\frac{10\times250^3}{12}\right] = (0.842\times10^6) + 1777\times(50+19.7)^2$

$$= 2[(13.021\times10^6) + (9.475\times10^6)] = 44.992\times10^6 mm^4$$

Since $L_{YY}$is less than I_{XX} , therefore the column will tend to buckle in Y-Y direction. Thus we shall take I equal to I_{YY} =44.992× $10^6$$mm^4$ (i.e., least of the two). Moreover as the column is fixed at its both ends, therefore equivalent length of the column.

$$L_e = \frac{l}{2} = \frac{6\times10^3}{2} = 3\times10^3\ mm$$

We know that least radius of gyration

$$k = \sqrt{\frac{I}{A}} = \frac{44.992\times10^6}{8554} = 72.5\ mm$$

∴ Rankine's crippling load on the column

$$P_R = \frac{\sigma_C.A}{1+a\left(\frac{L_e}{k}\right)^2} = \frac{320\times8554}{1+\frac{1}{7500}\times\left(\frac{3\times10^3}{72.5}\right)^2}$$

$$= 2228.5\times10^3 N = 2228.5kN$$

and safe load on the column = $\frac{Crippling\ load}{Factor\ of\ safety} = \frac{2228.5}{4}$ = 557.1 kN **Ans.**

4.2. COMBINED BENDING AND DIRECT STRESS

Direct and Indirect stresses – combination of stresses – Eccentric loads on Columns – Effects of Eccentric loads / Moments on Short columns – Combined direct and bending stresses – maximum and minimum stresses in sections – problems – Conditions for no tensions – limit of eccentricity – middle third rule – core or kern for square, rectangular and circular sections – chimneys subjected to uniform wind pressure – combined stresses n Chimneys due to self weight and Wind load – chimneys of hollow square and hollow circular cross sections only - problems

4.2. DIRECT STRESS

If a column is loaded with axial load *P,* then the column is subjected to direct stress and is given by

$\sigma_d = \frac{P}{A}$, where A is the cross-sectional area of the columnFigure 4.11 shows a direct stress.

4.2.1. COMBINED DIRECT AND BENDING STRESS

Figure 4.12 shows a column subjected to a load Pwhose line of action is at a distance '*e*' from the axis of the column. Apply two equal and opposite forces along the axis of the column as shown in Fig. 4.12(ii). Now the three forces acting on the column can be converted into two systems.

(i) An axial force P which will produce direct stress in the column,

as shown in Fig. 4.12 *(iii)*

(ii) Two equal and opposite forces forming a couple. The arm of the couple is '*e* '. The moment of the couple will be Px*e* and will produce bending stress (σ_b) in the column [Fig. 4.12 (iv)].

So a column Subjected to eccentric loading, is subjected to both direct and bending stress.

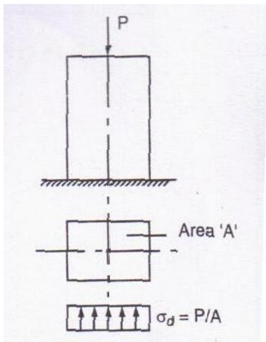


Fig4.11

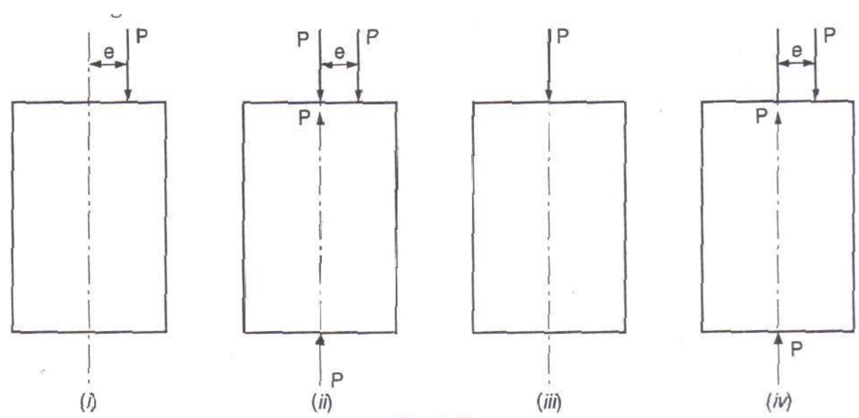


Fig4.12

Considered a column of width 'b' and depth 'd' subjected to an eccentric load P as shown in Fig.4.13.

Cross-sectional area, A=b x d.

Direct stress, $\sigma_d = \frac{P}{A} = \frac{P}{bd}$

Due to eccentricity of load, the column is subjected to B.M = P x e. this B.M. will produce bending stress in the column.

Bending stress, $\sigma_b = \frac{M}{Z}$

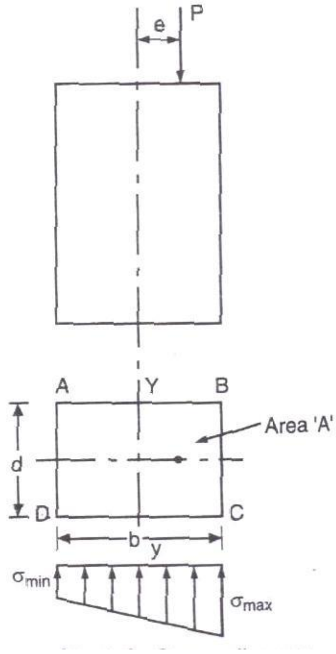
Where

Z = section modulus = $\frac{I}{Y}$

$$= \frac{db^2/12}{b/2} = \frac{db^2}{6}$$

(As the eccentricity of the load is from Y-Y axis, I = $\frac{db^3}{12}$)

$$\sigma b = \frac{P \times e}{\frac{db^2}{6}} = \frac{6Pe}{Ab}$$



The resultant stress at any point is the sum of direct and bending stress at that point.

Total stress $= \sigma_d \pm \sigma_b = \frac{P}{A} \pm \frac{M}{Z}$

Fig4.13

The + *ve*and-*ve*sign depends upon the position of the load. The stress will be maximum at the face *BC* as the load is near to *BC.* The stress will be minimum at the face *AD* as the load is away from *AD.*

$$\sigma_{max} = \frac{P}{A} + \frac{M}{Z} = \frac{P}{A} + \frac{6Pe}{Ab} = \frac{P}{A}\left[1 + \frac{6e}{b}\right]$$

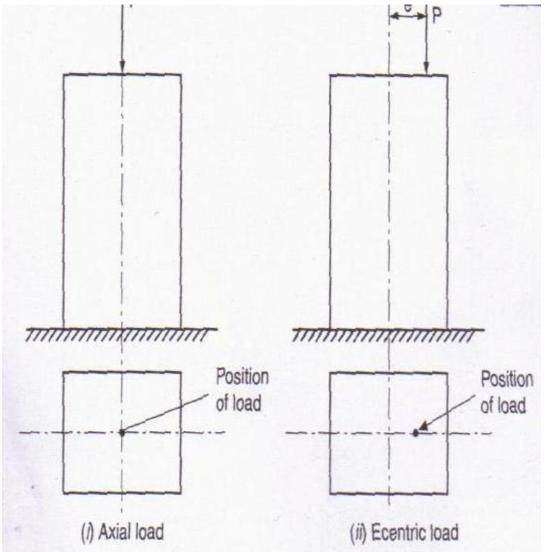
and $$\sigma_{min} = \frac{P}{A} + \frac{M}{Z} = \frac{P}{A} - \frac{6Pe}{Ab} = \frac{P}{A}\left[1 - \frac{6e}{b}\right]$$

$\sigma_{max}$is always compressive. If $\sigma_{min}$is +ve then it is compressive and if the value of σ_{min} is –ve, then it is tensile.

4.2.2. ECENTRIC LOAD

A load whose line of action passes through the center of gravity of the section, then the load is said to be axial load. A load whose line of action does not passes through the center of gravity of the section, then the load is called eccentric load.

Figure 4.14 (i) shows a section of column loaded with axial load P and Fig. 4.14 (ii) shows a section of column loaded with eccentric load. The distance between the line of action of load and axis of the column (passing through C.G.) is the eccentricity 'e' of the load. The eccentricity of a load may be about one axis or about both the axes.



4.2.3. LIMIT OF ECCENTRICITY

If a column is eccentrically loaded, then both direct and bending stress is simultaneously developed. If the direct stress (σd) is more than bending stress (σb), the stress in the section all through will be compressive. If the bending stress is more than direct stress, then there will be tensile stress. As the concrete columns are weak in tension, load should be applied in such a way that there is no tensile stress in the column. To avoid tensile stress, the bending stress (σb) should be less than or equal to direct stress (σd).

Hence,

$$\sigma_b \leq \sigma_d$$

$$\frac{M}{Z} \leq \frac{P}{A}$$

$$\frac{Pe}{Z} \leq \frac{P}{A}$$

$$e \leq \frac{Z}{A}$$

This shows that for tension in the column, eccentricity should be less than or equal to $\frac{Z}{A}$

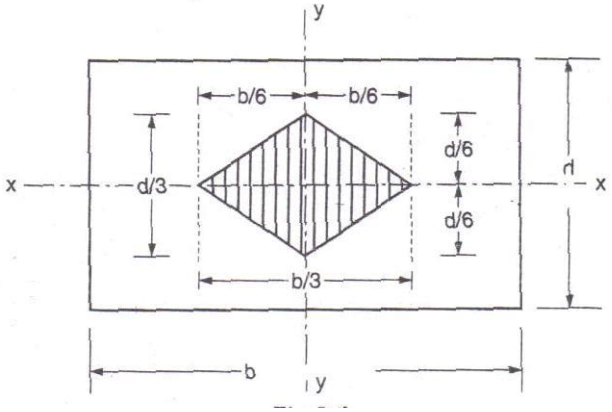
4.2.4. LIMIT OF ECCENTRICITY FOR A RECTANGULAR SECTION

Figure 4.15 shows a rectangular section of width 'b' and depth 'd'. Let the section be subjected to a load at a distance 'e', along x - x axis, from y - y axis. The bending will take place along y-yaxis.

$$I_{xx} = \frac{db^3}{12} \text{ and } y = b/2$$

Section modulus, $Z = \frac{I_{yy}}{Y}$

$$= \frac{db^3}{12} \times \frac{2}{b} = \frac{db^3}{6}$$



Area of cross section, A = bd.

For no tension, $e \leq \frac{Z}{A}$

$$e \leq \frac{db^2}{6bd}$$

$$e \leq \frac{b}{6}$$

Fig4.15

This means that for no tension, load can be applied on both sides of y-y at a distance b/6 on x-x axis as shown in Fig. 4.15.

Therefore, for no tension, the load must be placed with-in $\frac{b}{6} + \frac{b}{6} = \frac{b}{3}$ i.e., middle third of the width of section.

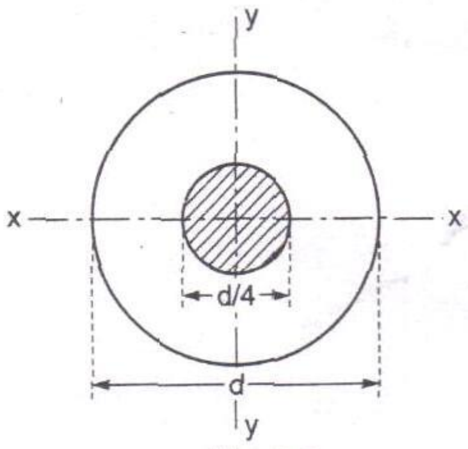
Similarly, for no tension, the load can be placed on y-y axis, on both sides of x-x axis, with in middle third of depth i.e., d/3.

If we join the four points on x-x and y-y axis, we get a rhombus. This rhombus is known as core or kernel as shown in Fig. 4.15. Thus, core or Kernel of the section is the area in which any eccentric load, if placed will not produce any tension in the section.

(ii) Limit of eccentricity for a circular section

Figure 4.16 shows a circular section of diameter'd'.

Let the section be subjected to a load at a distance 'e' from centroid on x-x-axis



$I_{xx} = I_{yy} = \frac{\pi d^4}{6^4}$

$Y = d/2$

Section modulus, $Z = \frac{I_{yy}}{Y}$

$= \frac{\pi d^4}{64} \times \frac{2}{d} = \frac{\pi d^3}{32}$

Area of cross-section, $A = \frac{\pi d^2}{4}$

For no tension

$e \leq \frac{Z}{A}$ Fig4.16

$e \leq \frac{\pi d^3}{32} \times \frac{4}{\pi d^2}$

$e \leq \frac{d}{8}$

This means that for no tension, the load can be eccentric, on either sides of centroid d/8 distance.

∴ Diameter of core $= 2X\frac{d}{8} = d/4$

Thus, for no tension in a circular section, the load must be placed within middle fourth of the section.

4.2.5. EFFECT OF ECCENTRIC LOADING

Consider a column subjected to an eccentric load 'w' at an eccentricity 'e' as shown that in fig.4.17 (a). the effect of eccentric load is equal to the effect of an axial load and a moment bending stress.

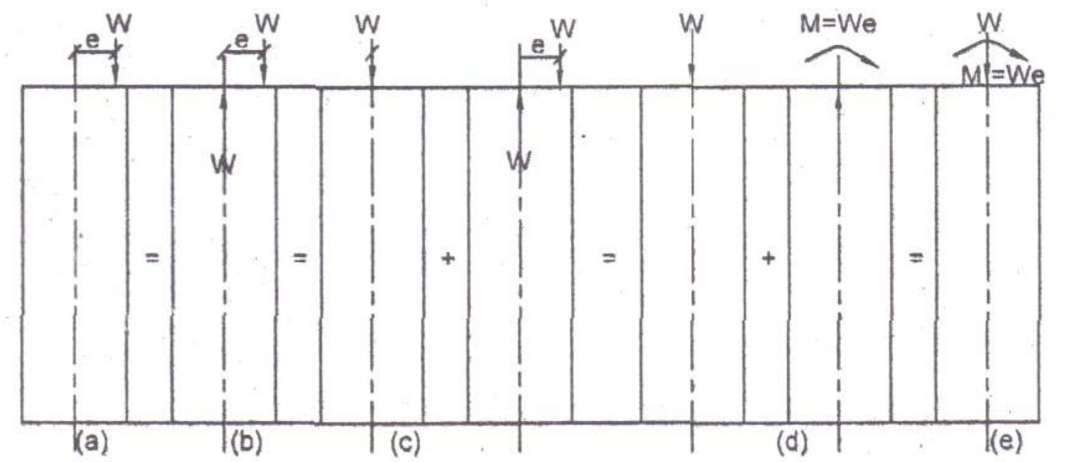


Fig4.17

The know the effect of eccentric load, W

(i) Introduce two equal and opposite loads, each of value W, along the axis of the 4.17 column.

(ii) Now, treat the whole system of loading as consisting of a statically equivalent algebraic system of loads as shown in fig.(c).

That is

a) A direct compressive force W along the longitudinal axis which produce direct compressive stress plus.

b) A couple due to eccentric load W and the other upward load W along the axis which produce the couple causes a BM, M = We which produce bending stresses.

(iii) Finally, the effect of eccentric load is reduced to consist of statically

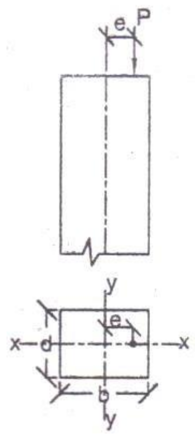
Equivalent centrally applied load or producing direct stress (σ_a) and a couple M=We producing bending stress (σ_a).

4.2.6. EFFECT OF ECCENTRIC LOADING ON SHORT COLUMN

Consider a short column of uniform cross section b x d subjected to an eccentric load 'P' acting on XX axis at an eccentricity 'e' from the Centroid of the section shown in fig 4.18

The axial load produces uniform direct compressive stress throughout the cross section. The moment due to couple produces bending stress.

Fig4.18



Direct compressive stress, σ_d = Load/Area

= P/A

Bending moment,

M = load * eccentricity

M= p*e

Bending stress,

$\sigma b = \pm M/I*y$

Where, I = moment of inertia about yy axis = db3/12

Y = distance of the fibre YY axis, $y_{max} = b/2$

$\therefore \sigma_b = \pm 12*Pe/db^3 * b/2 = \pm 6Pe/db^2$

Combined direct and bending stress

The total stress or resultant stress due to eccentric loading at any section is the algebraic sum of direct stress and bending stress.

$\sigma = \sigma_d + \sigma_b = P/A \pm M*y/I$

Maximum and minimum stresses

The maximum and minimum stresses are in the extreme fibres

$\sigma_{max} = \sigma_d + \sigma_b = P/A \pm M*y/I$

$\sigma_{min} = \sigma_d - \sigma_b = P/A - M*y/I$

for a rectangular column section,

A = b*d and m = P * e

$\sigma_{max} = \sigma_d + \sigma_b = P/A + 6M/db^2 = P/A + 6Pe/db^2$

$= P/A + 6 Pe/A*b = P/A [1+ 6e/b]$

$\sigma_{max} = P/A [1 + 6e/b]$

$\sigma_{min} = \sigma_d - \sigma_b = P/A - 6 Pe/A \times b = P/A[1 - 6e/b]$

$\sigma_{min} = P/A[1 - 6e/b]$

Maximum and minimum stress for a rectangular section,

σ_{max} = P/A[1 + 6e/b]

σ_{min} = P/A[1 + 6e/b]

When σ_{min} value is positive, there is compressive stress.

When σ_{min} value is negative, there is tensile stress.

When σ_{min} value is zero, there is no tension.

4.2.7. MIDDLE THIRD RULE

To avoid tension, the limiting value of eccentricity on either side of the geometric axis is b/6 and d/6 for rectangular sections and a/6 for square sections. This means if the load lies in the middle third portion, the section will be completely in compression.

The middle third rule states that, "when the point of application of the load lies within the middle third of the section, then the stress will be of compressive in nature throughout the section and there will be no tension anywhere in the section.

4.2.8. CHIMNEYS SUBJECTED TO UNIFORM WIND PRESSURE

Tall structures like chimneys, water tanks, towers are subjected to horizontal wind pressure on one side. It causes bending moment at the base. The bending moment reduces bending stress. Also the chimney has self-weighted and it produces direct or axial compressive stress at the base. The resultant stress at any section at the base of the chimney is the algebraic sum of bending stress due to wind pressure and axial stress due to self-weight.

Chimney may be square or circular in cross section. Sometimes the chimney may be tapered from large section at the bottom to a small section at the top.

The direct stress or axial stress, σ_d= W/A

The bending stress, σ_b= M/I*y (or) M/Z

Where,

W is the weight of chimney

A is the area of cross section

M is the bending moment due to horizontal wind pressure

I is the moment of inertia about YY axis

Y is the extreme fibre distance

Z is the section modulus, Z = I/y

The horizontal wind force on a unit area of a vertical plane is known as wind pressure. If the area exposed to wind pressure is curved, the magnitude of the force will be less than, when the area is a flat surface. The reduction factor 'k' is called as coefficient of wind resistance. Its value varies from 0.5 to 0.75. For cylindrical shafts k = 2/3, unless stated otherwise for square and rectangular chimney k = 1.

The total horizontal wind pressure, P = k*P_0*A_p

Where,

P = Total horizontal wind pressure

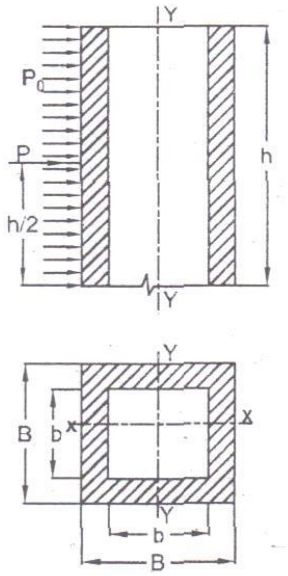
K = Coefficient of wind pressure

P_0 = Horizontal intensity of wind pressure

A_p = Projected area on which wind acts.

Maximum and minimum stress in square chimney

Consider a hollow square chimney of outer dimension B * B and inner dimensions b * b subjected to a horizontal wind pressure of intensity P_0 KN/m^2.



Let h be the height of the chimney.

Y be the unit weight of masonry.

Cross sectional area of chimney, = A_{ext} - A_{int}

A= (B * B) – (b * b)

Self-weight of chimney,

W = unit weight of masonry * Area * Height

W = γAh

∴ Direct stress, σ_d = W/A

Moment of inertia, I = 1/12 $[B^4-b^4]$

Extreme fibre density, y = B/2

Projected area, A_p = Bh

Total wind pressure, P = Coefficient of wind resistance

*Intensity * wind pressure * projected area

i.e., P = k*P_0*A_p

This pressure acts at h/2 from the base.

∴ Bending moment, M = P*h/2

∴ Bending stress, σ_b = M/I*y

Total stress $\sigma = \sigma_d + \sigma_b$

Fig4.19

Maximum stress, $\sigma_{max} = \sigma_d \pm \sigma_b$

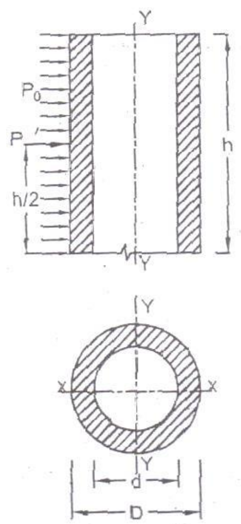
Minimum stress, $\sigma_{min} = \sigma_d - \sigma_b$

If the total stress is positive, it is compressive.

If the total stress is negative, it is tensile.

Maximum and minimum stress in circular chimney

Consider a hollow circular chimney of outer diameter 'D' and inner diameter'd' subjected to a horizontal wind pressure of intensity P_0 KN/m^2.



Let h be the height of the chimney.

γ be the unit weight of masonry.

Cross sectional area of chimney,

A = A_{ext}- A_{int}

= $\pi/4$ $[D^2 – d^2]$

Self-weight of chimney,

W = Unit weight of masonry * area* Height

W = γAh

∴Direct stress,

σ_d = w/A

Moment of inertia,

I = $\pi/64$ $[D^4 - d^4]$

Extreme fibre distance,

$y = D/2$

Projected Area,

$A_p = Dh$

Total wind pressure, P= Coefficient of wind resistance * intensity of wind pressure * Projected area

i.e., $P = k*P_0*A_p$

Fig4.20

This pressure acts at h/2 distance from the base.

K = 2/3 for circular sections.

Bending moment, $M = P * h/2$

$\therefore$ Bending stress, $\sigma_b = M/I * y$

Total stress, $\sigma = \sigma_d \pm \sigma_b$

Maximum stress, $\sigma_{max} = \sigma_d + \sigma_b$

Maximum stress, $\sigma_{min} = \sigma_d - \sigma_b$

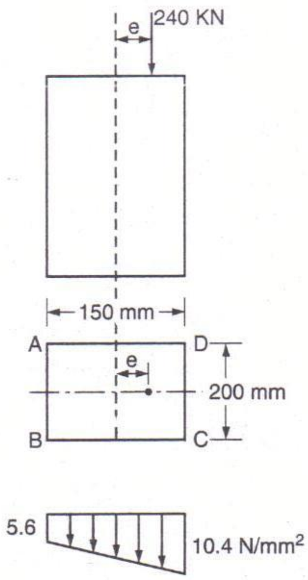
SOLVED EXAMPLES

EXAMPLE.4.17. A rectangular column of width 200 mm and of thickness 150 mm carries a point load of 240 KN at an eccentricity of 10 mm in a plane bisecting the thickness. Find the maximum and minimum stresses in the section.

Solution:

Given:

Point load,	P = 240 KN = 240000 N
Eccentricity,	e = 10 mm
Width,	b = 200 mm
Thickness,	d = 150 mm
Area	A = bd = 200 x 150 = 30000 mm^2



Let σ_{max} and σ_{min} use the maximum and minimum stress in the section

Using the relation

$\sigma_{max} = \frac{P}{A}\left(1+\frac{6e}{b}\right) = \frac{240000}{30000}\left(1+\frac{6\,x10}{200}\right)$

$= 8(1+0.3) \quad = 10.4$ N/mm^2 (comp)

$\sigma_{min} = \frac{P}{A}\left(1-\frac{6e}{b}\right) = \frac{240000}{30000}\left(1-\frac{6\,x10}{200}\right)$

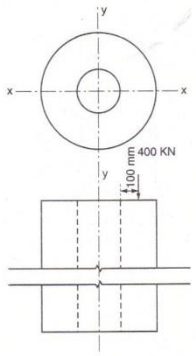
Fig4.21

= 8(1-0.3)

= 5.6 N/mm^2 (comp) **Ans**

EXAMPLE 4.18. A short column of hollow cylindrical section 250mm outside diameter and 150 mm inside diameter carries a vertical load of 400 KN along one of the planes 100 mm away from the axis of the column. Find the extreme stress intensities and state their nature.

Solution: Given Fig4.22



Inner diameter, d_i = 150 mm

Outer diameter, d_0 = 250 mm

Point load, P = 400 KN = 400000 N

Eccentricity, e = 100 mm

Distance of extreme fiber,

Y $=\frac{250}{2}$= 125 mm

Area, A= $\frac{\pi}{4}(d_0^2 - d_i^2) = \frac{\pi}{4} (250^2 - 150^2)$

= 31415.93 mm^2

Moment of inertia, I $=\frac{\pi}{64}(250^4 - 150^4)$

= 1.67×10^8 mm^4

Moment about the axis of the column,

= 4×10^7 N-mm

Section modules, Z $= \frac{I}{Y} = \frac{1.67\text{x}10^8}{125}$ = 1336000 mm^3

Using relation

$\sigma_{max} = \frac{P}{A} + \frac{M}{Z} = \frac{400000}{31415.93} + \frac{4\ X\ 10^7}{1336000}$ = 12.73 + 29.94

= 42.67 N/mm^2 (compressive) **Ans.**

$\sigma_{max} = \frac{P}{A} - \frac{M}{Z} = \frac{400000}{31415.93} - \frac{4\ X\ 10^7}{1336000}$ = 12.73 - 29.94

= 17.21 N/mm^2 (Tensile) **Ans.**

EXAMPLE 4.19. A short column of external diameter 400 mm and internal diameter 200 mm carries an eccentric load of 80 KN. Find the greatest eccentricity which the load can have without producing tension on the cross-section.

Solution:

Given

Inner diameter, d_i = 200 mm

Outer diameter, d_0 = 400 mm

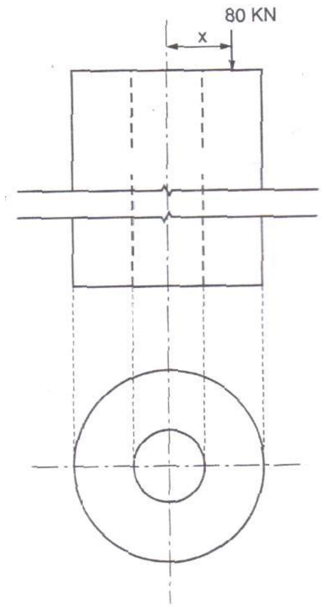
Point load, W = 80 KN = 8 x 1000= 8000 N

Let the greatest eccentricity be x mm which the load can have without producing tension.

Area, $A = \frac{\pi}{4}(d_0^2 - d_i^2) = \frac{\pi}{4}(400^2 - 200^2)$ Fig 4.23

$= 94247.78 \text{ mm}^2$

Moment of inertia,



$I = \frac{\pi}{64}(d_0^2 - d_i^2) = \frac{\pi}{64}(400^2 - 200^2)$

$= \frac{\pi}{64} \text{x} 2.4 \times 10^{10} = 1.18 \times 10^9 \text{ mm}^4$

$y = \frac{d0}{2} = \frac{400}{2} = 200 \text{ mm}$

Section Modulus,

$Z = \frac{I}{y} = \frac{1.18 \text{x} 10^9}{200} = 5.9 \times 10^6 \text{ mm}^2$

Moment about the axis of the column,

$M = P \times e = 80000x$ N-mm

For no tension

$\sigma_{max} = \frac{P}{A} - \frac{M}{Z} = 0$

$\therefore \quad \frac{P}{A} = \frac{M}{Z}$

or, $\frac{80000}{94247.78} = \frac{80000x}{5.9 \times 10^6}$

or, $\frac{80000 \times 5.9 \times 10^6}{94247.78 \times 80000} = x$

$x = \frac{5.9 \times 10^6}{94247.78} = 62.6$ mm **Ans.**

EXAMPLE 4.20. In a compression testing specimen 13 mm is diameter the line of thrust is parallel to the axis of the specimen but it displace from it. Determine the distance of the line of thrust from the axis, when the maximum stress is 15 % greater than the mean stress on a section normal to the axis.

Solution:

Given:

Diameter of specimen,

d = 13 mm

σ_{max} = 15 % greater than mean stress

Area, A $= \frac{\pi d^2}{4} = \frac{\pi(13)^2}{4} = 132.79 \text{ mm}^2$

Moment of inertia, I $= \frac{\pi d^4}{64} = \frac{\pi(13)^4}{64} = 1401.98 \text{ mm}^4$

Section modulus, Z $= \frac{I}{y} = \frac{1401.98}{6.5} = 215.69 \text{ mm}^3$

Let the eccentricity be *x* mm.

∴ Moment, M = Px

Mean stress = $\frac{\mathrm{P}}{\mathrm{A}} = \frac{\mathrm{P}}{132.73}$ N/mm^2

Maximum stress = Mean stress + 15% of mean stress

$\sigma_{max} = \frac{115}{100}$ x Mean stress

$\sigma_{max} = \frac{115}{100} \text{ x } \frac{P}{132.73}$ N/mm^2.....(i)

Using relation

$$\sigma_{max} = \frac{\mathrm{P}}{\mathrm{A}} + \frac{M}{Z}$$

$$\frac{115}{100 \text{ x } 132.73} = \frac{\mathrm{P}}{132.73} + \frac{\mathrm{P}x}{215.69} \qquad \text{[from(i)]}$$

or, $\frac{115}{100} = \frac{1}{132.73} + \frac{x}{215.69}$

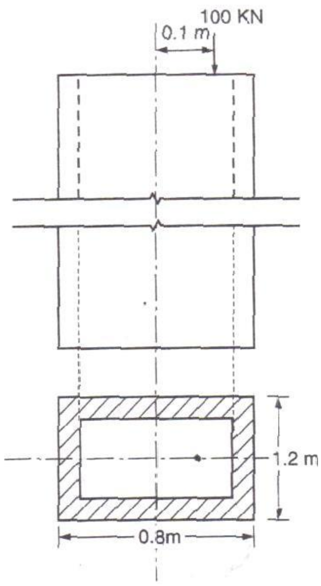
or, $\frac{115}{100} - \frac{1}{132.73} = \frac{x}{215.69}$

or, $0.0087 - 0.0075 = \frac{x}{215.69}$

$0.0012 \text{ x } 215.69 = x$

$x = 0.2588 \approx 0.26$ mm

EXAMPLE 4.21. A hollow rectangular masonry pier is 1.2 mm x 0.80 m, overall, the wall thickness being 0.15m. a Vertical load of 100 KN is transmitted in the vertical plane bisecting 1.2m side at an eccentricity of 0.1 m from the geometric axis of the section. Calculate the maximum and minimum stress intensities are the section.



Solution:

Given:

Point load, P = 100 KN = 100000 N

Eccentricity, e = 0.1 m = 100 mm

b_0 = 1200 mm

Outer diameter, d_0 = 800 mm

b_i = 1200 -150-150 = 900 mm

Inner diameter, d_i = 800-150-150 = 500 mm

Cross –sectional Area,

A = 1200× 800 – 900×500

= 960000 – 450000

Fig4.24

$= 510000\ mm^2$

Moment of inert

$$I = \frac{1}{12} \times 1200 \times 800^3 - \frac{1}{12} \times 900 \times 500^3$$

$= 5.12 \times 10^{10} – 9.37 \times 10^9$

$= 4.18 \times 10^{10}\ mm^4$

Moment,

$M = P \times e = 100000 \times 100$

$= 10 \times 10^6$ N-mm

Section modulus,

$$Z = \frac{I}{y} = \frac{4.18 \times 10^{10}}{400} = 10.45 \times 10^7\ mm^3$$

Using relation

$$\sigma_{max} = \frac{P}{A} + \frac{M}{Z} = \frac{100000}{510000} + \frac{10 \times 10^6}{10.45 \times 10^7}$$

$0.196 + 0.096 = 0.292\ N/mm^2$ **Ans.**

$$\sigma_{max} = \frac{P}{A} - \frac{M}{Z} = \frac{100000}{510000} - \frac{10 \times 10^6}{10.45 \times 10^7}$$

$= 0.196 - 0.096 = 0.1\ N/mm^2$ **Ans.**

EXAMPLE 4.21. A short hollow pier 1.5m square outside and 1 m square inside, supports a vertical point load of 7 KN located on a diagonal 0.8 m from the vertical axis of the pier. Neglecting the self weight of the pier. Calculate the normal stresses at the four outside corners on a horizontal section of the pier

Solution: Given:

Outer area, $A_0 = 1.5 \times 1.5$

$= 2.25\ m^2$

Inner area, $A_i = 1 \times 1 = 1\ m^2$

Point load, $P = 7kN$

Eccentricity, $e = 0.8m$

Area of cross section of pier = $A - A_t$

$= 2.25 – 1 = 1.25m^2$

Moment of inertia about the diagonal

$$I = 2 \times \frac{1}{12}(2.12 \times 1.06^3 - 1.41 \times 0.70^3)$$

$$= \frac{1}{6} \times (2.52 - 0.48) = 0.34m^4.$$

$$y = \frac{2.12}{2} = 1.06\ m$$

Section modulus,

$$z = \frac{I}{y} = \frac{0.34}{1.06} = 0.32\text{m}^3$$

Moment,

M= p × e

= 7×0.8 =5.6 KN-m

Direct stress,

$$\sigma_d = \frac{P}{A} = \frac{7}{1.25} = 5.6 \text{ KN/m}^2$$

Bending stress,

$$\sigma_b = \frac{M}{Z} = \frac{5.6}{0.32} = 17.5 \text{ KN/m}^2$$

At the corners 2 and 4 there will be no bending stress

∴Stress at cornet 1 = $\sigma_d + \sigma_b$=5.6 + 17.5

= 23.1KN/m^2 (Compressive)

Stress at corner 2 = σ_d =5.6 KN/m2 (Comp.)

Stress at corner 3 = σ_d -σ_b5.6 – 17.5

= -11.9 KN/m^2

= 11.9 KN/m^2 (Tensile)

Stress at corner 4 = σ_d =5.6 KN/m^2(Comp.)

EXAMPLE 4.22. A masonrypies of 3 m × 4 m supports a vertical load of 80KN as Shown in Figure 9.12.

(a) Find the stress developed at each corner of the pier.

(b) What additional load should be placed at the center of the pier,so that there Is no tension anywhere in the pier section?

(c) What are the stresses at the corners with The additional load in the center?

Solution. Given:

Width, b = 4 m

Depth, d = 3 m

Point load, p = 80 KN

Eccentricity from x-x axis

e_x= 0.5 m

Eccentricity from y-y axis is

e_y = 1 m

Area,

A = 4×3 = 12 m^2

Moment of inert about x-x,

$Ixx = \frac{1}{12} \times 4 \times 3^3 = 9\ m4$

Moment of inert about *y-y*,

$I_{yy} = \frac{1}{12} \times 3 \times 4^3 = 16\ m^4$

Moment about *xx* axis,

$M_x = 80 \times 0.5 = 40$ KN-m

Moment about *yy*axis,

$M_y = 80 \times 1 = 80$ KN-m

Direct stress,

$\sigma_d = \frac{P}{A}\frac{\mathbf{80}}{\mathbf{12}} = 6.67\ KN/m^2$

Bending stress due to e_x,

$\sigma_{bx} = \frac{Mx \times \mathbf{y}}{\boldsymbol{I}xx} = \frac{\mathbf{40 \times 1.5}}{\mathbf{9}} = 6.67\ KN/m^2$

Bending stress due to e_y,

$\sigma_{by} = \frac{My \times x}{\boldsymbol{I}yy} = \frac{\mathbf{80 \times 2}}{\mathbf{16}} = 10\ KN/m^2$

(a)

(*i*)Resultant stress at point *A*,

$\sigma_{\boldsymbol{A}} = \sigma_d - \sigma_{bx} + \sigma_{bym}$[... Point A is in 2nd quadrant]

$\sigma_{\boldsymbol{A}} = 6.67 - 6.67 + 10 = 10\ KN/m^2$

(*ii)* Resultant stress at point *B*,

$\sigma_{\boldsymbol{B}} = \sigma_d + \sigma_{bx} + \sigma_{by} = 6.67 + 6.67 + 10$

$= 23.34\ KN/m^2$(Compressive)

(*iii)*Resultant stress at point *C*,

$\sigma_{\boldsymbol{C}} = \sigma_d + \sigma_{bx} - \sigma_{by} = 6.67 + 6.67 - 10$

$= 3.34\ KN/m^2$(Compressive)

(*iv)*Resultant stress at point *D*,

$\sigma_{\boldsymbol{D}} = \sigma_d - \sigma_{bx} - \sigma_{by} = 6.67 - 6.67 - 10$

$= -10\ KN/m^2$(Compressive)

(b) Let *P*, be the additional load that should be placed in the pier section for no tension.

For no tension,

Compressive stress = Tensile stress

$\frac{P1}{A} = 10$

$P_1 = 10 \times A = 10 \times 10 = 120$ KN (c)stress due to additional load of 120 KN at the center

$= \frac{\mathbf{120}}{\mathbf{12}} = 10$ KN/m^2

(*i*) Stress at point *A*=10 + 10 = 20 KN/m2

*(ii)*Stress at point *B* = 23.34 + 10 = 33.34 KN/m^2

*(iii)*Stress at point *C = 3.34 + 10 = 13.34 KN/m^2*

*(iv)*Stress at point *D = - 10 + 10 = 0.*

EXAMPLE 4.23. A short column of rectangular cross-section 80mm ×60mm carries a load of 40 KN at a point 20 mm from the longer side and 35 mm from the shorter side. Determine the maximum compressive and Tensile stresses in the section.

Solution. Given:

Width, $b = 80$ m

Depth, $d = 60$m

Point load, $p = 40$ KN = 40000 N

$e_x = 10$ mm

$e_y = 5$ mm

Area,

$A = b \times d = 80 \times 60$

$= 4800$ m^2

Moment of inert about *x-x*,

$I_{xx} = \frac{\mathbf{1}}{\mathbf{12}} bd^3$

$= \frac{\mathbf{1}}{\mathbf{12}} \times 80 \times (60)^3 = 1.44 \times 10^6$ mm^4

Moment of inert about *y-y*,

$I_{yy} = \frac{\mathbf{1}}{\mathbf{12}} bd^3 = \frac{\mathbf{1}}{\mathbf{12}} \times 60 \times (80)^3 = 2.56 \times 10^6$m4

$y = \frac{\mathbf{60}}{\mathbf{2}} = 30$ mm

$x = \frac{\mathbf{80}}{\mathbf{2}} = 40$ mm

Moment about *x-x* axis,

$M_x = P \times e_x = 40000 \times 10 = 400000$ N-mm

Moment about *y-y* axis,

$M_y = P \times e_y = 40000 \times 5 = 200000$ N-mm

Direct stress, $\sigma_d = \frac{P}{A} \frac{40000}{4800} = 8.33$ N/mm^2

Bending stress due to e_x, $\sigma_{bx} = \frac{M\mathrm{x} \times \mathbf{y}}{\boldsymbol{I}\mathrm{xx}} = \frac{400000 \times 30}{1.44 \times 10\ 6}$

$= 8.33$ N/mm^2

Bending stress due to c_y,

$$\sigma_{by} = \frac{My \times x}{Iyy} = \frac{200000 \times 40}{2.56 \times 10\ 6}$$

$= 3.125\ N/mm^2$

Maximum stress = $\sigma_d + \sigma_{bx} + \sigma_{by}$ = 8.33 + 8.33 + 3.125

$= 19.78\ N/mm^2$

Maximum stress = $\sigma_d - \sigma_{bx} - \sigma_{by}$ = 8.33 - 8.33- 3.125

$= -3.125\ N/mm^2$

$= 3.125\ N/mm^2$(Tensile)

EXAMPLE 4.24. A short column of diameter' D' and internal diameter' d' carries ecentric load' W' find the greatest ecentricity which the load can have without producing tension on the cross-section of the column.

Solution: Given :

Outer diameter = *D*

Inner diameter = *d*

Point load, *p* = *w*

Let the greatest eccentricity for no tension be *e.*

Cross-sectional area, $A = \frac{\pi}{4}(D^2 - d^2)$

Moment of inertia, $I = \frac{\pi}{64}(D^4 - d^4)$

Moment, $M = W \times e$

$Y = \frac{D}{2}$

Direct stress, $\sigma_d = \frac{P}{A} = \frac{W}{\frac{\pi}{4}(D4 - d4)}$

Bending stress, $\sigma_b = \frac{My}{I} = \frac{W \times e \times D/2}{\frac{\pi}{64}(D4 - d4)}$

For no tension

Direct stress = Bending stress

$$\frac{W}{\frac{\pi}{4}(D2 - d2)} = \frac{W \times e \times D/2}{\frac{\pi}{64}\ D4 - d4 \times 2}$$

$$\frac{4W}{\frac{\pi}{4}(D2 - d2)} = \frac{32We \times D}{\pi(D2 + d2)(D2 - D2)}$$

Dividing both sides by $\frac{4W}{\pi(D2 - d2)}$, we get $1 = \frac{8eD}{D2 + d2}$

Or, $\frac{D2 + d2}{8D} = e$

…Greatest eccentricity,

$e = \frac{D2 + d2}{8D}$ **Ans.**

EXAMPLE 4.25. A column section 300 mm external diameter and 150 mm internal diameter support an axial load of 2600 KN and an eccentric load of P KN at an eccentricity of 400 mm. if the compressive and tensile stresses are not to exceed 149 N/mm^{2}and 60 N/mm^2 respectively, find the magnitude of the load P .

Solution: Given :

Outer diameter, d_o = 300 mm

Inner diameter, d_i = 150mm

Axial load W = 2600 kn = 2600 $\times$1000 N

Eccentric load = 400 mm

Cross-sectional area,

$A = \frac{\pi}{4}(do^2 - di^2) = \frac{\pi}{4}(300^2 - 150^2)$

=53014.38 mm^2

Moment of inertia,

$I=\frac{\pi}{64}(do^2 - di^2) = \frac{\pi}{64}(300^4 - 150^4)$

$= \frac{\pi}{64} \times 7.59 \times 10^9 = 37.26 \times 10^7 mm^4.$

$Y = \frac{300}{2}$ = 150 mm

Sectional modulus, $Z = \frac{I}{Y} = \frac{37.27 \times 107}{150}$

= 2484666.67 mm^3

Moment,

$M= P\times \boldsymbol{e} = P\times 400$

=400P KN-mm

=400000 PN-mm

Direct stress,

$\sigma_d=\frac{(2600+ P)\times \mathbf{1000}}{53014.38}$ = 0.019 (2600+P)

Bending stress,

$\sigma_b=\frac{400000P}{\mathbf{2484666.67}}$ =0.16 P N/mm^2

$\sigma_{max}= \sigma_{d} + \sigma_b$

Or

140 = 0.019 (2600 + P) + 0.16 P

140 =49.4 + 0.019P + $0.16P$

140 – 49.4 = 0.179 P

… 90.6 = 0.179P

… $p = \frac{90.6}{0.179}$ = 506.15 KN …(*i*)

$\sigma_{min} = \sigma_b - \sigma_d$

60= 0.16*P*- 0.019 (2600 + *p)*

=*0.16P* - 49.4 -0.019*P*

60 + 49.4 = 0.141 *P*

90.6 = 0.179*P*

… *p*= 775.89KN

The magnitude of *P* will be minimum of (*i*) and (*ii*)

i.e., *P* = 506.14 KN Ans.

EXAMPLE 4.26. A square chimney 24 m high has an opening of 1 m × 1 m and wall thickness 0.25 m, calculate tha maximum stress in the masonry if the horizontal wind pressure is 2000 N/m^2 and masonry weight 20000 N/m^3

Solution: Given :

Area of opening = 1 m ×1 m

Wall thickness = 0.25 m

Wind pressure, *P* = 2000 N/m

Unit weight of masonry,

γ = 20000 N/m^3

Height of chimney, h = 24 m

Cross-sectional area of chimney,

A = (1.5 ×1.5) - (1 × 1)

= 1.25m^2

Weight of chimney, W = γ*A*h

Direct stress,

$\sigma_d = \frac{W}{A} = \frac{\gamma A h}{A}$

=γ*h*= 20000 ×24

= 480000 N/m^2

Wind load,

P = p× 1.5 × 24

= 2000 ×1.5×24

=72000 N

Moment about the base of the chimney $= P \times \frac{h}{2}$

=72000 ×12 = 864000 N-m

Moment of inertia, $I = \frac{1}{12}(1.5^4 - 1^4) = \frac{1}{12} \times 4.06 = 0.338 \text{ m}^4$.

$Y = \frac{1.5}{2} = 0.75$ m

Bending stress, $\sigma_b = \frac{My}{I} = \frac{864000 \times \mathbf{0.75}}{0.338}$

$= 1917159.8$ N/m^2

$\sigma_{max} = \sigma_{b+} \sigma_d = 480000 + 1917159.8$

$= 2397159.8$ N/m^2 (Compressive) Ans.

EXAMPLE 4.27. A masonry chimney of a hollow circular section has external and internal diameters of 2.5 m and 2 m respectively. It is subjected to uniform horizontal wind pressure of 1800 N/m^2 .determine the maximum height of chimney for developing no tension at the base. Density of masonry is 20000 N/m^3 .

Solution : Given :

d_o= 2.5 m

d_i= 2.0 m

p =1800 N/m^2

γ = 20000 N/m^3

Let the height of chimney be hm .

Weight of chimney = γAh

Direct stress,

$\sigma_d = \frac{\gamma Ah}{A} = \gamma h = 20000\, h$ N/m^2

Wind load,

$P = p \times 2.5 \times h$

$= 1800 \times 2.5 \times h = 4500\, h$ N

Moment due to P about the base

$M = P \times e = 4500\, h \times \frac{h}{2}$

$= 2250\, h^2$ N-m

Moment of inertia, $I = \frac{\pi}{64} \times (2.5^4 - 2^4) = 1.132$ m^4

$Y = \frac{2.5}{2} = 1.25$ m

Section modulus, $Z = \frac{I}{y} = \frac{1.132}{1.25} = 0.906$

Bending stress, $\sigma_b = \frac{M}{Z} = \frac{2250h2}{0.906}$

$= 2483.44\, h^2$ N/m^2

For no tension $\sigma_d = \sigma_b$

20000 h = $2483.4\, h^2$

$\frac{20000}{2483.4} = h = 8.05$ m Ans.

5. DAMS AND RETAINING WALL

5.1 MASONRY DAMS

Gravity Dams – Derivation of Expression for maximum and minimum stresses at Base – Stress distribution diagrams – Problems – Factors affecting Stability of masonry dams – Factor of safety- Problems on Stability of Dams– Minimum base width and maximum height of dam for no tension at base – Elementary profile of a dam – Minimum base width of elementary profile for no tension.

5.1. MASONRY DAMS

5.1.1. Introduction

Dam is a Massive structure constructed by concrete or masonry to retain water. A massive wall constructed across the river to store the water is also called Dam or Masonry dam.

1.Rectangular section dam.

2.Trapezoidal section with water face vertical.

3.Trapezoidal section with water face inclined.

The water stored on the u/s side of the dam exerts the horizontal water pressure.

The following forces are acting on the gravity dam.

1.Selfweight of dam (W).

2.Horizontal water pressure (P).

3.Uplift pressure at the base.

4.Due to horizontal water thrust and self weight of dam, the resultant thrust (R) will hits the base.

5.1.2. Derivation for maximum and minimum stresses

Consider a trapezoidal section masonry dam as shown in fig.

Let

a = Top width

b = bottom width

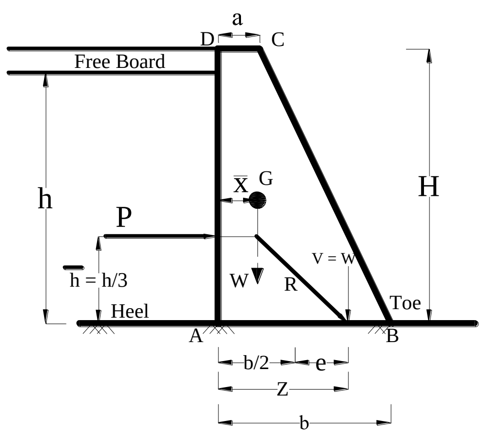
H = Height of dam

h = Depth of water

γ = Specific weight of masonry

ω = Specific weight of water

Consider 1m length of dam



i. Weight of dam/m (W)

W = γ x volume = γ x A x ℓ

W = $\gamma \dfrac{(a+b)}{2}$ x H x 1 = $\gamma \dfrac{(a+b)}{2}$ H

$$\boxed{W = \gamma \frac{(a+b)}{2} H} \text{ kN} \quad \ldots\ldots\ldots\ldots\ldots \rightarrow (1)$$

This will act at a distance of $\bar{x}$ from vertical face

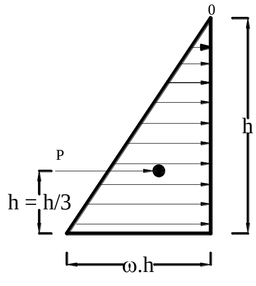
$$\bar{x} = \frac{a^2 + ab + b^2}{3\,(a+b)}$$

ii. Horizontal water pressure/m (P)

The intensity of pressure at the top is zero, and at bottom is ‘ωh’.

Draw pressure distribution diagram as shown in fig.

Total water pressure = area of pressure diagram



Pressure Diagram

$$P = \frac{1}{2} \times (\omega h) \times h = \frac{\omega h^2}{2}$$

$$\boxed{P = \frac{\omega h^2}{2}} \text{ kN} \quad \ldots\ldots\ldots\ldots\ldots \quad \rightarrow (2)$$

This will act at the c.g. of pressure diagram.

ie $\bar{h} = \frac{h}{3}$ from base

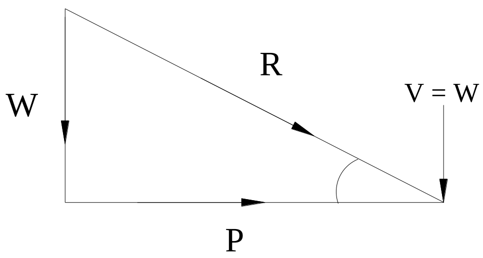
Total vertical force at the base (V) = (W)

Resultant thrust (R)

The combined effect of water pressure and self weight of dam produces the resultant thrust ‘R’

$$R = \sqrt{V^2 + P^2} = \sqrt{W^2 + P^2}$$

This resultant thrust cuts the base at a distance ‘Z’ from vertical face (A).



Position of resultant thrust (Z)

Taking moment about A

$$(v \times z) = W\bar{x} + P\bar{h}$$

$$W \times Z = W\bar{x} + P\bar{h}$$

$$\therefore \boxed{Z = \bar{x} + \frac{P}{W}\bar{h}} \qquad (V = W)$$

Eccentricity (e)

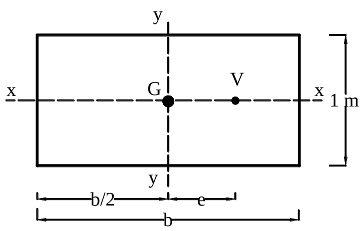
The eccentricity of resultant thrust is ‘e’

$$e = (Z - b/2)$$

Stresses at the base of dam

Consider in one meter length of wall.

Area A = b x 1 = b x 1



Z = Section modulus about yy axis

$$\therefore Z = Z_y = \frac{db^2}{6} = \frac{1 \times b^2}{6} = \frac{b^2}{6}$$

M = Moment due to eccentric load

$$M = W \times e$$

$$\sigma_c = \frac{W}{A} \text{ direct compressive stress}$$

$$\sigma_b = \frac{M}{Z} = \frac{W \times e}{Z} = \text{ Bending stress}$$

Total stress at the base (σ)

$$\sigma = \sigma_c \pm \sigma_b$$

$$\sigma = \left(\frac{W}{A} \pm \frac{We}{Z}\right) = W\left[\frac{1}{b \times 1} \pm \frac{e}{b^2/6}\right]$$

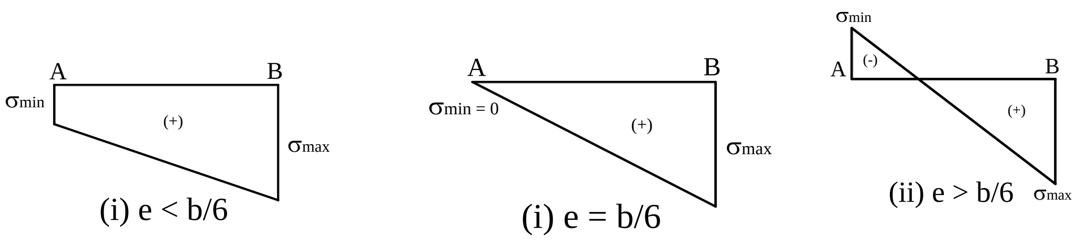
$$\sigma = \frac{W}{b}\left(1 \pm \frac{6e}{b}\right)$$

$$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right) \text{ at toe}$$

$$\sigma_{min} = \frac{W}{b}\left(1 - \frac{6e}{b}\right) \text{ at heel}$$

5.1.3. Stress diagram at the base

The stress distribution diagram is drawn according to the maximum and minimum stresses as given below.



5.1.4. Factors affecting the stability of masonry dams

The following are the causes of failures of masonry dam.

1. Tension at the base
2. Sliding along the base
3. Overturning about the toe
4. Crushing of masonry at the base.
5. Uplift pressure

5.1.4.1. Conditions for stability of masonry dams

1. To avoid tension at the base
2. To avoid sliding of the wall along the base
3. To avoid overturning of dam
4. To avoid crushing of masonry at the base.
5. To avoid uplift pressure at the base.

1. Conditions to avoid tension at the base

To avoid tension at any where, in the dam, the minimum stress (σ_{min}) should be greater than or equal to zero.

ie $\sigma_{min} \geq 0$

$$\frac{W}{b}\left(1-\frac{6e}{b}\right) \geq 0 \quad ; \quad \left(1-\frac{6e}{b}\right) \geq 0 \; ; 1 \geq \frac{6e}{b}$$

$$\frac{6e}{b} \leq 1 \qquad \boxed{e \leq \frac{b}{6}}$$

ieTo avoid tension the resultant force must cut the base within the middle third of the base.

Note: We know,

The position of resultant thrust from heel is 'Z'.

$$Z = \frac{b}{2}+e=\frac{b}{2}+\frac{b}{6}=\frac{2}{3}b$$

$$Z \leq \frac{2}{3} b$$

ie To avoid tension

$$e \leq \frac{b}{6} \; (or) \; Z \leq {}^{2}\!/_{3}\, b$$

2. Conditions to avoid sliding

To prevent sliding of the wall along the base, the total frictional force (μ x W) should be greater than total horizontal water pressure (P).

$$\therefore (\mu \text{ x } W) > P$$

$$\left.\begin{array}{l}\text{Factor safety}\\ \text{against sliding}\end{array}\right\}=\frac{\text{Total frictional force}}{\text{Total horizontal water pressure}}$$

$$\text{F.S. (Sliding)} \; \frac{\mu \text{ x } W}{P} \geq 1.00$$

Where

μ = Coefficient of friction

For design purpose the F.S. (Sliding) is considered as 1.5.

3. Conditions to prevent overturning

The horizontal water pressure (P) may tend to overturn about the toe.

$$\left[\; \bar{h} = {}^{h}\!/_{3} \right]$$

To prevent overturning of dam, the stabilizing moment should be greater than the overturning.

$$\left.\begin{array}{l}\text{Factor of safety}\\ \text{against overturning}\end{array}\right\}=\frac{\text{Balancing moment}}{\text{Overturning moment}} \geq 1.0$$

$$\text{F.S. (overturning)} = \frac{W(b-\bar{x})}{P\bar{h}} \geq 1.0$$

For design purpose, the factor of safety against overturning is taken as 1.5 to 2.0.

4. Conditions to prevent crushing at the base

The maximum compressive at the base σ_{max} = $\frac{W}{b}\left(1+\frac{6e}{b}\right)$

To avoid crushing at the base

Maximum compressive stress should be less than the safe bearing capacity of soil (or) allowable compressive stress.

ie $\sigma_{max} \leq$ SBC of soil (or) allowable compressive stress

5. Condition to avoid uplift pressure

To avoid uplift pressure at the base of the dam, the weight of dam should be greater than uplift pressure $W > \omega.b.h$

Where

W = Total weight of dam at base

$(\omega \times h)(b \times 1) = \omega.b.h$ = Total uplift pressure.

5.1.5. Minimum base width for no tension

The bottom width (b) of a dam is calculated using the conditions of stability of dam.

(i) To avoid tension at the base

The eccentricity should be with in the middle third.

ie $e \leq \frac{b}{6}$; $Z \leq \frac{2}{3}b$

$b \geq 6e$,

(ii) To avoid sliding of dam

$$\left.\begin{array}{l}\text{Factor of safety}\\ \text{(sliding)}\end{array}\right\} = \frac{\text{Frictional force}}{\text{Horizontal water pressure}} \geq 1.5$$

$$\text{F.S. (Sliding)} = \frac{\mu \times W}{P} \geq 1.5$$

(iii) To avoid overturning of dam

$$\text{Factor of safety overturning} = \frac{\text{Balancing moment}}{\text{Overturning moment}} \geq 1.5$$

$$\text{F.S. (overturning)} = \frac{W(b-\bar{x})}{P\bar{h}} \geq 1.5 \text{ (or) } 2$$

(iv) To avoid crushing at the base

The allowable compressive stress should be greater than maximum compressive stress.

SBC of soil or $\sigma_{allowable} \geq \sigma_{max}$

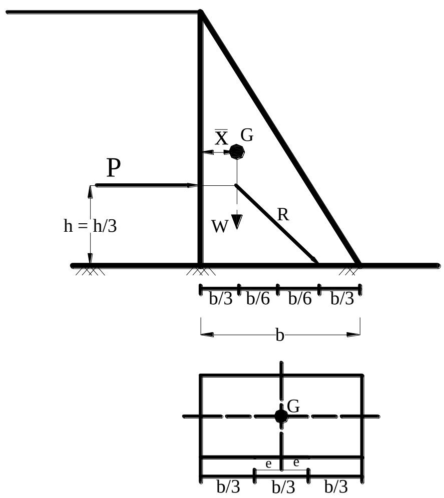
The minimum base width required is the maximum value of base width for the above conditions.

5.1.6. Maximum height of dam for no tension

The height dam is calculated for the stability conditions.

The maximum height of the dam is the minimum height value for the stability conditions.

5.1.7. Elementary profile of a dam



To avoid tension, the resultant force (R) must cut the base with in the middle third.

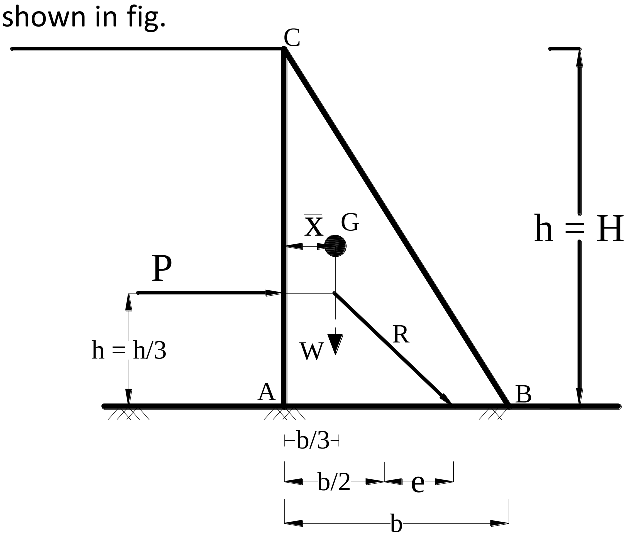
$$e = \frac{b}{6} \text{ on either side of geometrical axis.}$$

This must be satisfied for both dam full and dam empty.

The right angled triangle cross section of a dam will be satisfied the middle third rule when the dam is full or empty, and is called elementary profile of a dam.

5.1.8. Minimum base width of elementary profile for no tension

Consider a triangular section masonry dam of bottom width 'b' and height 'H'. Retains water on its vertical face to full depth (h = H) as shown in fig.



Let

γ = Specific weight of masonry

ω = Specific weight of water

$S = \left(\frac{\gamma}{\omega}\right)$ = Relative density.

Consider 1m length of wall.

$$W = \gamma \times \left(\frac{1}{2} \times b \times H\right) \times 1 = \frac{\gamma b H}{2}$$

$$\bar{x} = b/3$$

$$P = \omega\, h^2/2 = \frac{\omega H^2}{2},\ \bar{h} = H/3$$

Taking moment about A

$$W \times Z = W\,\bar{x} + P\bar{h}$$

$$Z = \left(\bar{x} + \frac{P}{W}\,\bar{h}\right)$$

$$Z = \left(\frac{b}{3} + \frac{\omega H^2/2}{\left(\frac{\gamma b H}{2}\right)} \times H/3\right) = \left(\frac{b}{3} + \frac{\omega H^2}{3\gamma b}\right)$$

To avoid tension

$Z = \frac{2}{3} b$

$$\therefore \frac{2}{3} b = \left(\frac{b}{3} + \frac{\omega H^2}{3 \gamma b} \right) = \frac{1}{3} \left(b + \frac{\omega H^2}{\gamma b} \right)$$

$$2b = b + \frac{\omega H^2}{6 \gamma b}$$

$$(2b - b) = \frac{\omega H^2}{\gamma b} = \frac{H^2}{\left(\frac{\gamma}{\omega}\right) b} = \frac{H^2}{S x b}$$

$$b^2 = \frac{H^2}{S} \qquad \left(\therefore S = \frac{\gamma}{\omega}\right)$$

$$b = \sqrt{H^2/S} = \frac{H}{\sqrt{S}} \quad ; \quad \boxed{(b) = \frac{H}{\sqrt{S}}}$$

The minimum base width for a right angled triangular section

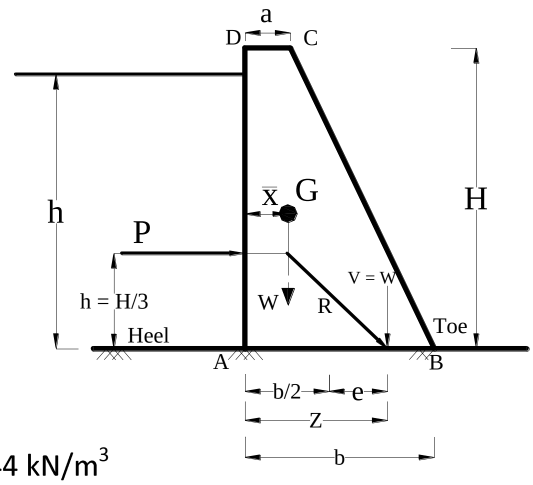
$$b = \frac{H}{\sqrt{S}}.$$

Note: Practically the trapezoidal cross section is provided.

Problem 5.1

A trapezoidal section masonry dam 8m height, 1.5m wide at top and 3.5m wide at base, retains water on its vertical face to height of 7.5m if the relative density of masonry is 2.4. determine the stress intensities at base and draw stress distribution diagram.

Given



Top width	a	= 1.5m
Bottom width	b	= 3.5m
Height of dam	H	= 8m
Depth of water	h	= 7.5m
Specific gravity	S	= 2.4 ;

Assume unit wt. of water ω = 9.81 kN/m^3

Unit weight of masonry, γ = S x ω

γ = 2.41 x 9.81 = 23.544 kN/m^3

Required

$$\sigma = \frac{W}{b}\left(1 \pm \frac{6e}{b}\right) = ?$$

Solution

Consider 1 m length of dam

i. Weight of dam 1m (W)

$$W = \gamma \frac{(a+b)}{2} H$$

$$W = 23.544 \left(\frac{1.5+3.5}{2}\right) 8$$

W = 470.88 kN

This will act at $\bar{x}$ from vertical face

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$$

$$\bar{x} = \frac{1.5^2+1.5\,x\,3.5+3.5^2}{3(1.5+3.5)}$$

$\bar{x}$ = 1.316 m

ii. Total horizontal water pressure / m (P)

$$P = \frac{\omega h^2}{2} = \frac{9.81\,x\,7.5^2}{2} = 275.90\,kN$$

This will act at $\bar{h}$ from base.

$$\bar{h} = h/3 = \frac{7.5}{3} = 2.5m$$

iii. Position of resultant pressure (Z)

$$Z = \bar{x} + \frac{P}{W}\bar{h}$$

$$Z = 1.316 + \left(\frac{275.9}{470.88}\,x\,2.5\right) = 2.78m$$

$$\boxed{Z = 2.780m}$$

iv. Eccentricity (e)

$$e = (Z - b/2) = \left(2.78 - \frac{3.5}{2}\right) = 1.038m$$

v. Stresses at the base of dam (σ)

$$\sigma = \frac{w}{b}\left(1 \pm \frac{6e}{b}\right)$$

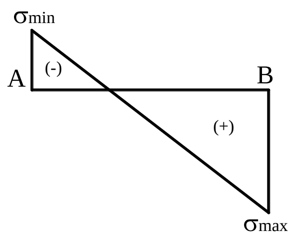
$$\sigma_{max} = \frac{w}{b}\left(1 + \frac{6e}{b}\right) \text{ at toe (B)}$$

$$\sigma_{max} = \frac{470.88}{3.5}\left(1 + \frac{6\,x1.03}{3.5}\right) = 372.09 \text{ kN/m}^2 \text{ (Comp.)}$$

$$\sigma_{min} = \frac{w}{b}\left(1 - \frac{6e}{b}\right) \text{ at heel (A)}$$

$$\sigma_{min} = \frac{470.88}{3.5}\left(1 - \frac{6\,x1.03}{3.5}\right) = -\,103.017 \text{ kN/m}^2 \text{ (Tension)}$$

Stress diagram

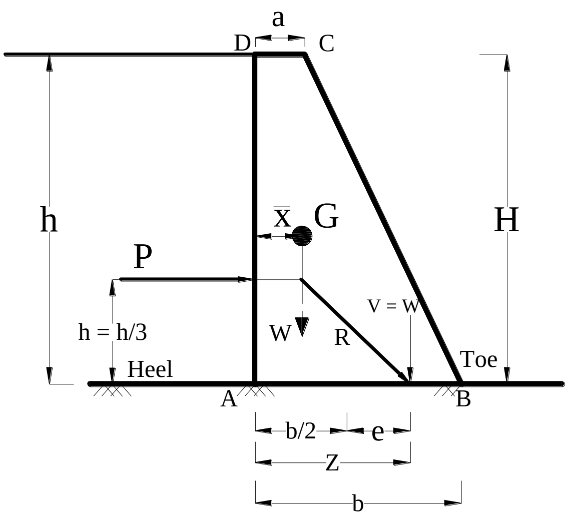


Problem 5.2

A masonry dam of trapezoidal section 2m wide at top 6m wide at base and 12m height retains water on its vertical face, unit weight of masonry and water are 23 kN/m^3 and 9.81 kN/m^3 respectively. Determine the stresses at the base of dam.

a. **When the dam is full**

b. **When the dam is empty**



Given

Top width a = 2m

Bottom width b = 6m

Heath H = 12m

Unit weight of water ω= 9.81 kN/m^3

Unit weight of masonry γ = 23 kN/m^3

Required

$$\sigma = \frac{w}{b}\left(1 \pm \frac{6e}{b}\right) = ?$$

Solution

i. Weight of dam 1m (W)

$$W = \gamma \frac{(a+b)}{2} H = 23 \left(\frac{2+6}{2}\right) x\ 12 = 1104 \text{ kN}$$

This will act at $\bar{x}$ distance from vertical face

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} = \frac{2^2+(2x6)+6^2}{3(2+6)} = 2.16\text{m}$$

a) <u>When the dam is full (h = H = 15 m)</u>

$$P = \frac{\omega h^2}{2} = \frac{9.81x12^2}{2} = 706.32 \text{ kN}$$

This will act at $\bar{h}$ from base.

$$\bar{h} = \frac{h}{3} = \frac{15}{3} = 4 \text{ m}$$

iii. Position of resultant thrust (Z)

$$Z = \bar{x} + \frac{P}{W}\bar{h} = 2.16 + \frac{706.32}{1104} x\ 4 = 4.72 \text{ m}$$

iv. Eccentricity

$$e = (Z - b/2) = (4.72 - 6/2) = 1.72 \text{ m}$$

v. Stress at the (σ)

$$\sigma = \frac{w}{b}\left(1 \pm \frac{6e}{b}\right)$$

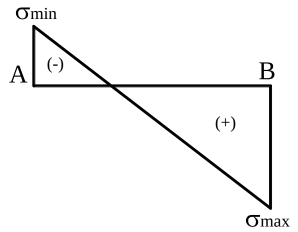
$$\sigma_{max} = \frac{w}{b}\left(1 + \frac{6e}{b}\right) = \frac{1104}{6}\left(1 + \frac{6x1.72}{6}\right) \text{ at toe (B)}$$

σ_{max} = 500.48 kN/m^2 (Comp.)

$$\sigma_{min} = \frac{w}{b}\left(1 - \frac{6e}{b}\right) \text{ at heel (A)} = \frac{1104}{6}\left(1 - \frac{6x1.72}{6}\right)$$

σ_{min} = – 132.48 kN/m^2 (Tension)

Stress diagram



b. <u>When the dam is empty (</u>h = 0)

Horizontal water thrust/m (P)

$$P = \frac{\omega h^2}{2} = \frac{9.81 x 0^2}{2} = 0$$

$$Z = \bar{x} + \frac{P}{W}\,\bar{h} \qquad [P = 0]$$

$$Z = \bar{x} = 2.16 \text{ m}$$

Eccentricity(e)

$$e = Z - b/2 = (2.16 - 6/2) = -0.84 \text{ m}$$

Stress at the base (σ)

$$\sigma = \frac{w}{b}\left(1 \pm \frac{6e}{b}\right)$$

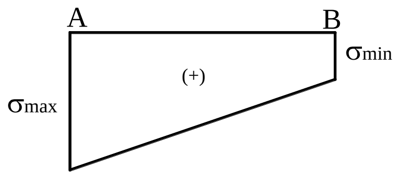
$$\sigma_{max} = \frac{w}{b}\left(1 + \frac{6e}{b}\right) = \frac{1104}{6}\left(1 + \frac{6 \text{ x} (-0.84)}{6}\right)$$

$$\sigma_{max} = 29.44 \text{ kN/m}^2 \text{ (Comp.)}$$

$$\sigma_{min} = \frac{w}{b}\left(1 - \frac{6e}{b}\right) = \frac{1104}{6}\left(1 - \frac{6 \text{ x} (-0.84)}{6}\right)$$

$$\sigma_{min} = 338.56 \text{ kN/m}^2 \text{ (Comp.)}$$

Stress diagram



Problem 5.3

A trapezoidal dam 1.5m wide at top 3.5m wide at base and 8m height retains water on its vertical face with free board of 0.5m. Specific gravity of masonry is 2.4. Check the stability of dam coefficient of friction 0.6 Max. allowable stress = 300 kN/m^2.

Given

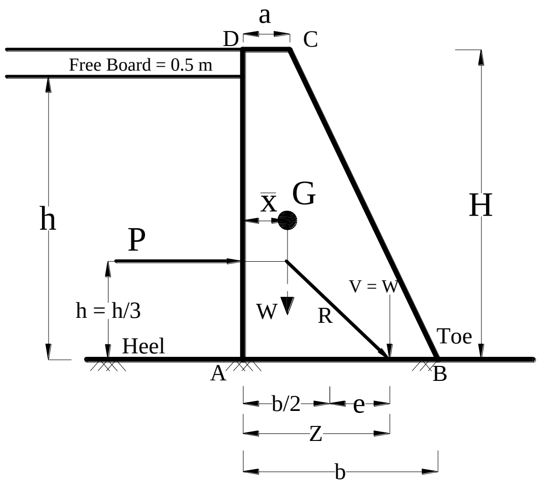
Top width a = 1.5m

Bottom with b = 3.5m

Height H = 8 m

Free board = 0.5 m

Depth of water h = 8-0.5 = 7.5 m



$\frac{\gamma}{\omega}$ = S = 2.4

γ = 2.4 x 9.81 = 23.54 kN/m^3

ω = 9.81 kN/m^3

Solution

i. Weight of dam 1m (W)

$$W = \gamma \frac{(a+b)}{2} H = 23.54 \left(\frac{1.5+3.5}{2}\right) \times 8$$

W = 470.88 kN

This will act at $\bar{x}$ from vertical face

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} = \frac{1.5^2+(1.5x3.5)+3.5^2}{3(1.5+3.5)}$$

$\bar{x}$ = 1.316m

ii. Total horizontal pressure / 1m (p)

$$P = \frac{\omega h^2}{2} = \frac{9.81x7.5^2}{2} = 275.90 kN/m$$

This will act at $\bar{h}$ from base.

$$\bar{h} = h/3 = \frac{7.5}{3} = 2.5m$$

iii. Position of resultant pressure (Z)

$$Z = \bar{x} + \frac{P}{W}\bar{h} = 1.316 + \frac{275.9}{470.88} \times 2.5$$

Z = 2.78m

iv. Eccentricity (e)

e = (Z – $b/2$) = (2.78 – $3.5/2$) = 1.03m

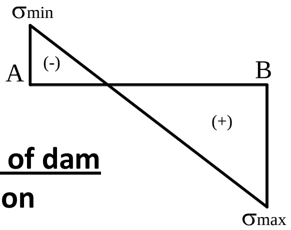
v. Stress at the base (σ)

$$\sigma = \frac{w}{b}\left(1 \pm \frac{6e}{b}\right)$$

$$\sigma_{max} = \frac{w}{b}\left(1+\frac{6e}{b}\right) = \frac{470.58}{3.5}\left(1+\frac{6x1.03}{3.5}\right) = 372.09 \text{ kN/m}^2 \text{ (Comp.)}$$

σ_{min} $= \frac{w}{b}\left(1 - \frac{6e}{b}\right) = \frac{470.88}{3.5}\left(1 - \frac{6x1.03}{3.5}\right)$ = – 103.017 (Tension) kN/m^2

Stress diagram



Check for stability of dam

i. To avoid tension at base

σ_{min} ≥ 0

e $\leq b/6$ (or) Z $\leq 2/3$ b

e = 1.03m

$b/6 = 3.5/6 = 0.58$

e > $b/6$ Hence tension will develop at base.

ii. To avoid sliding

F.S $= \frac{\mu w}{P}$ = 1.0, $= \frac{0.6 x 470.38}{275.9}$ = 1.02 > 1.00

F.S = 1.02 > 1.0 Hence safe against sliding

iii. To avoid failure from over turning

F.S $= \frac{w(b-\bar{x})}{P\bar{h}}$ = 1.0, $= \frac{470.88(3.5-1.30)}{275.9 x 2.5}$ = 1.48 > 1.00

1.48 > 1.0 (Safe)

Hence safe against overturning.

iv. To avoid from crushing

SBC > σ_{max}

SBC = 300 kN/m^2

σ_{max} = 372.06 kN/m^2

SBC < σ_{max}

Hence not safe.

Problem 5.4

A trapezoidal dam 4m high has top width of 1m, with vertical face exposed of water it retain water up to its top level. Find the min. base width required. To avoid tension and sliding. Take unit wt. masonry as 22 kN/m^3 and that of water as 9.81 kN/m^3. Take μ = 0.6 and F.S. = 1.5

Given

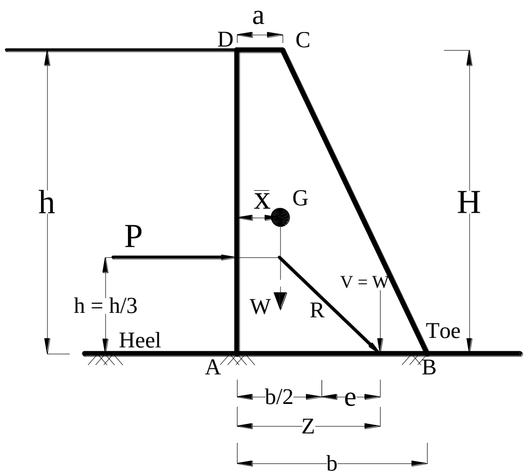
Top width a = 1m

Height of dam H = 4m

Depth of water $h = H = 4$ m

$\gamma = 22$ kN/m^3

$\omega = 9.8$ kN/m^3



Condition : No tension at base

Required

b = ?

Solution

i. Weight of dam / m (W)

$$W = \gamma \frac{(a+b)}{2} H = 22 \left(\frac{1+b}{2}\right) \text{x } 4$$

$$\boxed{W = 44\,(1 + b)} \quad \text{.................. (1)}$$

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} = \frac{1^2 + 1xb + b^2}{3\,(1+b)}$$

$$\boxed{\bar{x} = \frac{b^2 + b + 1}{3(1+b)}} \quad \text{.................. (2)}$$

ii. Total horizontal water thrust / 1m (p)

$$P = \frac{\omega h^2}{2} = \frac{9.81 x 4^2}{2} = 78.48\,kN$$

$$\bar{h} = h/3 = (4/3)$$

iii. Position of resultant pressure

$$Z = \bar{x} + \frac{P}{W}\,\bar{h} = \left[\frac{b^2 + b + 1}{3(1+b)}\right] + \left[\frac{78.48}{44(1+b)} \text{x} \frac{4}{3}\right]$$

$$Z = \frac{b^2 + b + 1}{3(1+b)} + \frac{7.13}{3(1+b)}$$

$$Z = \frac{b^2 + b + 8.13}{3(1+b)} \quad \text{......... (3)}$$

Bottom width required

a) To avoid tension at base

$e \leq b/6$ (or) $Z \leq 2/3$ b

$$\frac{2}{3}\,b = \frac{b^2+b+8.13}{3(1+b)}$$

$2b\,(1 + b) = b^2 + b + 8.13$

$2b + 2b^2 = b^2 + b + 8.13$

$b^2 + b - 8.13 = 0$ .... (4)

$$b = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 8.13}}{2 \times 1} = 2.4 \text{ m}$$

$$\boxed{b_1 = 2.40\text{m}}$$

b) To avoid sliding

$$\text{F. S. (sliding)} = \frac{\mu \, x \, w}{P} \geq 1.50$$

$$= \frac{0.6\, x\, 44\,(1+b)}{78.48}$$

$$= 1.5$$

$0.6 \times 44\,(1+b) = (1.5 \times 78.48)$

$26.40\,(1+b) = 117.72$

$$1 + b = \frac{117.72}{26.40} = 4.46$$

$b = 4.46 - 1 = 3.46$ m

$$\boxed{b_2 = 3.46 \text{ m}}$$

Minimum bottom width required (b) = Maximum of b_1 and b_2

Result:

b = 3.46 m

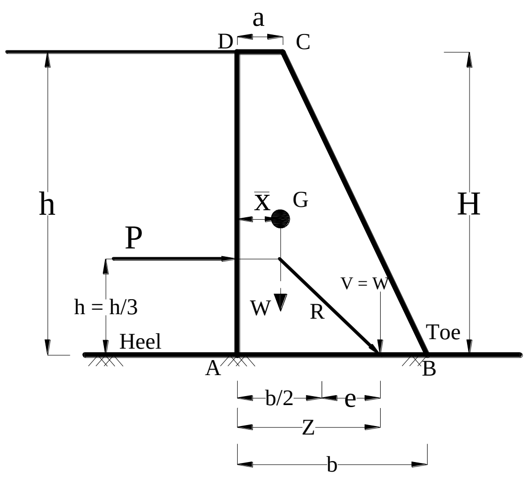
Problem 5.5

A masonry dam 1m wide at top 3m wide at base retains water on its vertical face, the dam is full. Determine the max height of dam required.

- **i. For no tension at base**
- **ii. To avoid sliding, F.S. against sliding = 1.5**
- **iii. To avoid overturning,F.S. over turning = 2.0**

Take μ = 0.60 Unit weight of masonry = 23 kN/m^3 and Unit weight of water= 9.81 kN/m^3

Given



a = 1m

b = 3m

μ = 0.6

ω = 9.81 kN/m^3

γ = 23 kN/m^3

F.S (Sliding) = 1.5

F.S. (Overturning) = 2

Required

H = h = ?

Solution

i. Weight of dam 1m (w)

W $= \gamma \dfrac{(a+b)}{2}$ H = 23 $\dfrac{(1+3)}{2}$ H = 46 H .... (1)

$\bar{x}$ $= \dfrac{a^2+ab+b^2}{3(a+b)} = \dfrac{1^2+(1+3)+3^2}{3(1+3)} =$ 1.16m

ii. Total horizontal thrust / 1m (P)

P $= \dfrac{\omega h^2}{2} = \dfrac{9.81 x H^2}{2} =$

P = 4.905 H^2 kN .... (2)

$\bar{h}$ = $H/3$

iii. Position of resultant thrust (Z)

Z = $\bar{x} + \dfrac{P}{W}\bar{h}$

Z = 1.16 + $\dfrac{4.905 H^2}{46 H}$ x $H/3$ = (1.16 + 0.0355H^2) .... (3)

Maximum height

i. To avoid tension

Z ≤ $2/3$ b = $2/3$ x 3

Z = (2/3)x3 m = 2 m

(1.08 + 0.0355H^2) = 2

0.0355 H^2 = 2 – 1.16 = 0.84

H $= \sqrt{\dfrac{0.84}{0.0355}} =$ 4.86 m $\boxed{H_1 = 4.86m}$

ii. To avoid sliding

F.S. (sliding) = 1.5

F.S. $= \frac{\mu w}{P} = \frac{0.6 x 46.4}{4.905 H^2} = 1.5$

H $= \frac{0.6 x 46}{4.905 x 1.5} = 3.75$ m

$\boxed{H_2 = 3.75m}$

iii. To avoid overturning

F.S (overturning) = 2.0

F.S $= \frac{W(b-\bar{x})}{P\bar{h}} = 2.0$ m

$= \frac{(46xH)(3-1.08)}{4.905 H^2 x (H/3)} = 2.0$

$= \frac{46(3-1.08)3}{4.905 x \mathrm{H}^2} = 2.0$

$H^2 = \frac{46(3-1.08)3}{4.905 x 2} = 27$; H = $\sqrt{27}$ = 5.19 m

$\boxed{H_3 = 5.19m}$

Maximum height required = min of H_1, H_2 & H_3

Result:

H = 3.75 m

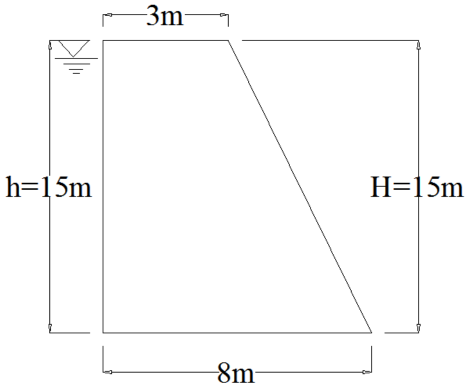
Additional Problems

Problem:1

A concrete dam trapezoidal section of 15m height retains water on its vertical face to its full height. The top width of the dam is 3m and the bottom is 8m, weight of concrete is 24KN/m^3. Find i) the resultant thrust on the base per meter length of dam .ii) the point where the resultant cuts the base. iii) intensities of maximum and minimum stresses at the base.

Given data:

Height of the dam , H=15m
Height of water, h =15m
Top width, a = 3m
Bottom width, b = 8m
Weight of concrete, γ_m= 24KN/m^3
Specific weight of water, γ_w = 9.81KN/m^3



Solution:

Maximum stress, $\sigma_{max} = \frac{w}{b}\left(1 + \frac{6e}{b}\right)$

consider 1m length of dam

Width of dam w = $\left(\frac{a+b}{2}\right) \times \gamma_m \times H$

$= \left(\frac{3+8}{2}\right) \times 24 \times 15 = 1980$ kN

w= 1980 KN acting at $\bar{x}$ from the vertical face.

Eccentricity :

$e = z - \frac{b}{2}$

$Z = \bar{x} + \left(\frac{p \times h}{w \times 3}\right)$

$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$

$= \frac{3^2+(3 \times 8)+8^2}{3(3+8)}$

$\bar{x}$ =2.94m

Water pressure ,

$P = \frac{\gamma_m \times h^2}{2}$

$= \frac{24 \times 15^2}{2}$

P = 2700 kN

$Z = 2.94 + \left(\frac{2700 \times 15}{1980 \times 3}\right)$

Z = 9.76m.

Eccentricity :

$e = z - \frac{b}{2}$

$= 9.76 - \frac{8}{2}$

e = 5.76m.

Maximum stress ,

$\boldsymbol{\sigma}_{\mathbf{max}} = \frac{w}{b}\left(1 + \frac{6e}{b}\right)$

$= \frac{1980}{8}\left(1 + \frac{6\,(5.76)}{8}\right)$

σ_{max} =1316.7 kN/m^2(compression)

Minimum stress ,

$\boldsymbol{\sigma}_{\mathbf{mini}} = \frac{w}{b}\left(1 - \frac{6e}{b}\right)$

$= \frac{1980}{8}\left(1 - \frac{6\,(5.76)}{8}\right)$

σ_{mini} = 821.7kN/m^2 (compression)

Resultant thrust

$R = \sqrt{P^2 + w^2}$

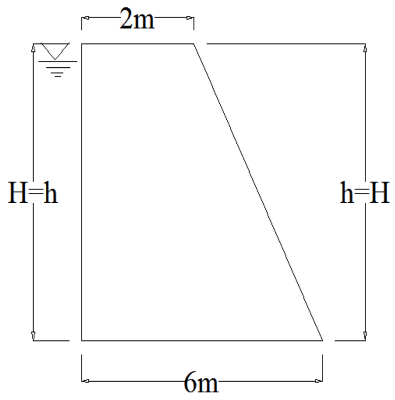
$= \sqrt{2700^2 + 1980^2}$

R= 3348.2 kN

Problem:2

A trapezoidal masonry dam 2m wide at top and 6m wide at base has to retain water on its vertical face up to top. Calculate maximum height of dam to ensure that no tension is developed at the base. take weight of masonry as 20 kN/m^3 and weight of water as 9.81 kN/m^3.

Given data:



Top width, a=2m
Bottom width, b=6m
Height of dam, H= height of water, h
Specific weight of masonry, γm= 20 KN/m3
Weight of water, γw =9.81 KN/m3

To find:

Maximum height of the dam

Solution:

Consider 1m length of wall

Lateral water pressure ,

$P = \frac{\gamma_m x\ H^2}{2}$

$P = \frac{9.81x\ H^2}{2}$ = 4.905 H^2

Weight of dam ,

w = $(\frac{a+b}{2})$ x γ_m x H

= $(\ \frac{2+6}{2})$ x 20 x H

w = 80H

$\overline{x}= \frac{a^2+ab+b^2}{3(a+b)}$

= $\frac{2^2+(2x\ 6)+6^2}{3(2+6)}$

$\overline{x}$ =2.17 m

Z = $\overline{x}$ + ($\frac{Pxh}{wx3}$)

Z = 2.17 + ($\frac{4.905\ H^2xH}{80Hx3}$)

Z = 2.17 +0.02H^2 --------------------(1)

For no tension condition

e = ($\frac{b}{6}$)

e = z - ($\frac{b}{2}$)

Z= e + ($\frac{b}{2}$)

Z = ($\frac{b}{6}$) + ($\frac{b}{2}$)

Z = ($\frac{2b}{3}$)

Z = ($\frac{2(6)}{3}$)

Z =4 m -------------------------(2)

Equating (1) & (2)

Z =2.17+0.02 H^2

4 =2.17+0.02 H^2

H^2= ($\frac{1.87}{0.02}$) = 91.5 m

H = 9.56 m.

Problem :3

A masonry dam 8m high retains water to a depth 7m. The top width of the dam is 1.5m and bottom width 5m. The relative density of masonry is 2.4. Calculate the normal stress intensities at base of dam. Sketch the distribution.

Given data:

Height of dam, H=8m

Height of water, h=7m

Top width of dam, a=1.5 m

Bottom width of dam, b=5m

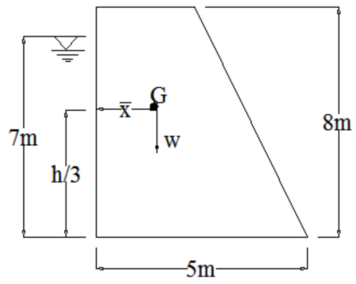
Relative density of masonry,$\rho = 2.4$

To find:

Normal intensities at the base

i.e., σ_{max} and σ_{min}

Solution:



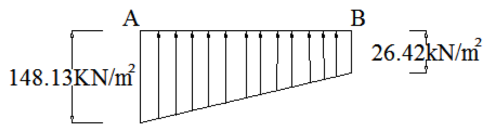
Relative density of masonry $= \frac{\text{sp.wt of masonry}}{\text{sp.wt of water}}$

i.e., $\rho = \frac{\gamma_m}{\gamma_w}$

Take γ_m= 9.81 kN/m^3

$2.4 = \frac{\gamma_m}{9.81}$

γm=23.54 kN/m3



STRESS DISTRIBUTION DIAGRAM

Consider 1m length of wall

Lateral water pressure ,

$P = \frac{\gamma_m x\, h^2}{2}$

$P = \frac{9.81 x\, 7^2}{2}$

P = 240.35 kN

Weight of dam,

$w = (\frac{a+b}{2})$ x γ m x H

$= (\frac{1.5+5}{2})$ x 23.54 x 8

W = 612.04 kN

$\overline{x} = \frac{a^2+ab+b^2}{3(a+b)}$

$= \frac{1.5^2+(1.5x\,5)+5^2}{3(1.5+5)}$

$\overline{x}$ =1.78m.

$Z = \overline{x} + (\frac{Pxh}{wx3})$

$Z=1.78+ (\frac{240.35x7}{612.04x3})$

Z=2.69 m.

Eccentricity (e),

$e = z- (\frac{b}{2})$

$= 2.69 - (\frac{7}{2})$

e = - 0.81m. (hogging)

Maximum stress

$\sigma_{max} = \frac{w}{b} (1 + \frac{6e}{b})$

$= \frac{612.04}{7} (1 + \frac{6(0.81)}{7})$

σ_{max} =148.13 kN/m^2

Minimum stress

$\sigma_{mini} = \frac{w}{b} (1- \frac{6e}{b})$

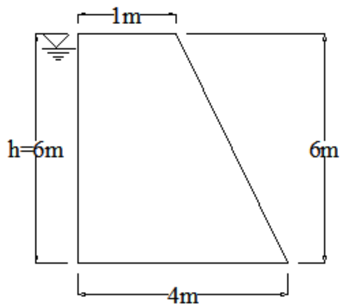
$= \frac{612.04}{7} (1- \frac{6(0.81)}{7})$

σ_{min} =26.72 kN/m^2.

problem :4

A masonry dam 1m wide at the top, 4m wide at the base and 6m height with the water side is vertical. The water stored up to the top of wall. Find the maximum and minimum normal intensities at the base, it's the specific weight of masonry is 22kN/m^3. Calculate also the normal stress intensities when reservoir empty.

Given data:



Top width, a=1m

Bottom width, b=4m

Height of dam, H=6m

Height of water, h=6m

Specific weight of masonry ,γ_m=22 KN/m^3

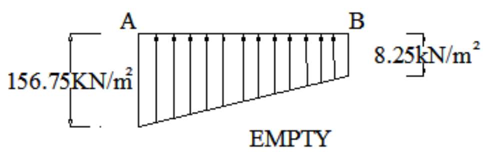
Specific weight of water ,γ_w=9.81 KN/m^3

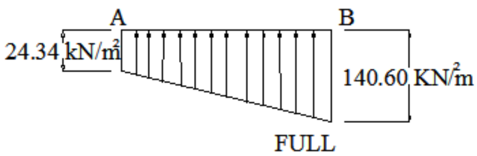
To find :

1.Maximum stress ,σ_{max}

2.Minimum stress ,σ_{mini}

3.Normal stress intensity when reservoir empty





Solution:

Consider 1m length of wall

Weight of dam,

$w = (\frac{a + b}{2})x \gamma_m x H$

$= (\frac{1 + 4}{2}) x22x 6$

=330 KN

$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$

$= \frac{1^2+(1 \text{x} 4)+4^2}{3(1+4)}$

$\bar{x}$=1.4m

Lateral water pressure ,

$P = \frac{\gamma_m \text{ x } h^2}{2}$

$= \frac{9.8 \text{ x } 6^2}{2}$

=176.58 kN

Z = $\bar{x}$ + ($\frac{\text{P x h}}{\text{w x 3}}$)

Z =1.40 + ($\frac{176.58 \text{ x } 6}{330 \text{ x } 3}$)

Z=2.47 m

Eccentricity(e) ,

e = z - ($\frac{b}{2}$)

= 2.47 - ($\frac{4}{2}$)

e = 0.47m.

(i) When the reservoir is full:

Maximum stress

$\sigma_{max} = \frac{w}{b} (1 + \frac{6e}{b})$

$= \frac{330}{4} (1+ \frac{6(0.47)}{4})$

σ_{max} =140.66 kN/m^2

Minimum stress

$\sigma_{mini} = \frac{w}{b} (1 - \frac{6e}{b})$

$= \frac{330}{4}(1- \frac{6(0.47)}{4})$

σ_{mini} =24.34 kN/m^2

(ii) When the reservoir is empty

Lateral water pressure,

P = 0

Weight of dam ,

w= 330 kN

$\bar{x}$=1.40m

z= $\bar{x}$=1.40m

e = z- ($\frac{b}{2}$)

e = 1.40 - ($\frac{4}{2}$)

= - 0.6 m

Maximum stress

$\sigma_{max} = \frac{w}{b} (1+ \frac{6e}{b})$

$= \frac{330}{4} (1 + \frac{6 (0.6)}{4})$

σ_{max} =156.75 KN/m^2**(compression)**

Minimum stress

$\boldsymbol{\sigma_{mini}} = \frac{w}{b} (1- \frac{6e}{b})$

$= \frac{330}{4} (1 - \frac{6 (0.6)}{4})$

σ_{mini} = 8.25 KN/m^2**(compression)**

answer:

i)When the reservoir is full:

$\boldsymbol{\sigma_{max}}$= 140.66 KN/m^2 (compressive)

$\boldsymbol{\sigma_{mini}}$= 24.34 KN/m^2 (compressive)

ii)When the reservoir is empty

$\boldsymbol{\sigma_{max}}$= 156.75 kN/m^2 (compressive) @A

$\boldsymbol{\sigma_{mini}}$= 8.25 kN/m^2 (compressive) @B

Problem:5

A trapezoidal masonry dam 1.5m width at top and 5m wide at the base .it is 8.0m height with a vertical water face and retains water face and retains water to a depth of 7.5m. find the maximum and minimum stress intensities at the base. take weight masonry as 22 kN/m^3 and weight of water as 9.81 kN /m^3.

Given data

Top width , a = 1.5m

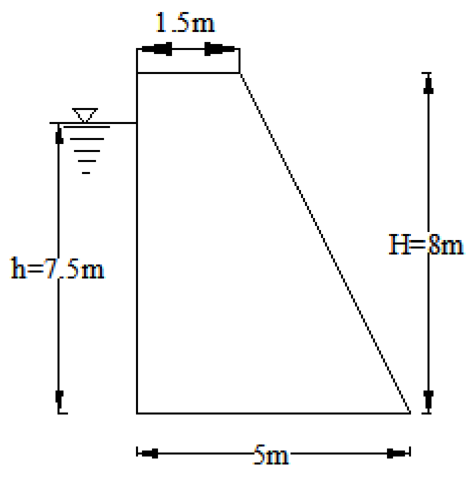
Bottom width, b = 5m

Height of dam, H = 8m

Height of dam, h = 7.5m

Specific weight of masonry, γ_m= 22kN/m^3

Specific weight of water, γ_w= 9.81 kN/m^3



To Find

Max. and min. stresses at the base.

Solution

Consider 1m length of wall.

Lateral water pressure,

$P = \frac{\gamma_w . h^2}{2}$

$= \frac{9.81 \times 7.5^2}{2}$ =275.91 kN

Weight of dam,

$W = \frac{(a+b)}{2} \times \gamma_m \times H$

$= \frac{(1.5+5)}{2} \times 22 \times 8 = 572.00$ kN

$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$

$= \frac{1.5^2+(1.5x5)+5^2}{3(1.5+5)} = 1.782$ m

$Z = \bar{x} + (\frac{P}{W} \times \frac{h}{3})$

$=1.782 + (\frac{275.91}{572} \times \frac{7.5}{3})$

Z = 2.99 m

$e = z - \frac{b}{2}$

$=2.99 - \frac{5}{2} = 0.49$ m

Max stress,

$\sigma_{max} = \frac{W}{b}(1 + \frac{6e}{b})$

$= \frac{572}{5}(1 + \frac{6(0.49)}{5})$ = **181.67 kN/m²(comp)**

Min stress,

$\sigma_{min} = \frac{W}{b}(1 - \frac{6e}{b})$

$= \frac{572}{5}(1 - \frac{6(0.49)}{5})$ = **47.13 kN/m² (comp)**

Result:

(i) Max. stress, σ_{max} = 181.67 kN/m²(comp)

(ii) Min .stress, σ_{min} = 47.13 kN/m² (comp)

Problem: 6

A masonry dam 20 m high retains water to a depth of 18m.The top width of the dam is 5m and bottom width is 15m. The relative density of masonry is 2.4.Calculate the normal stress intensities at the base of the dam. Sketch the stress distribution.

Given data

Height of dam, H = 20 m

Height of water, h = 18 m

Top width of dam, a = 5m

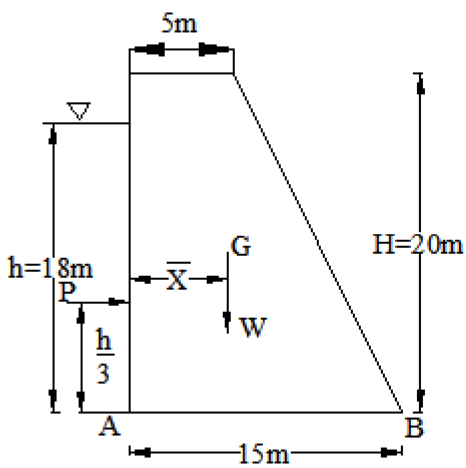
Bottom width of dam, b = 15m

Relative density of masonry, ρ = 2.4

To Find:

Normal stress intensities at the base.

i.e., σ_{max} and σ_{min}



Solution

Relative density of masonry = $\frac{\text{specific weight of masonry}}{\text{specific weight of water}}$

i.e.,$\rho = \frac{\lambda}{\gamma_w}$

Take γ_w =9.81 kN/m³

i.e., $2.4 = \frac{\lambda}{9.81}$

$\therefore \lambda = 2.4 \times 9.81 = 23.54$ kN/m³

Consider 1m length wall

Lateral water pressure,

$$P = \frac{\gamma_w h^2}{2}$$

$$= \frac{9.81 x 18^2}{2} = 1589.22kN$$

Weight of dam,

$$W = \frac{(a+b)}{2} \text{ x } \gamma_m \text{ x } H$$

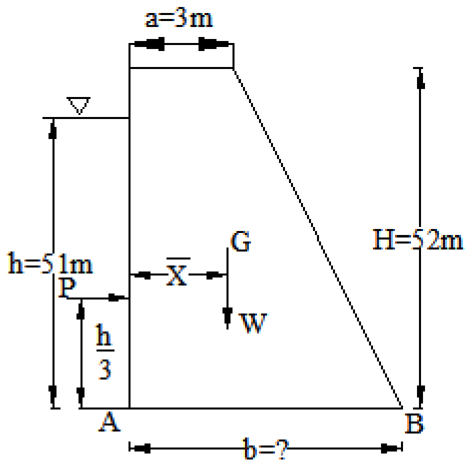
$$= \frac{(5+15)}{2} \text{ x } 23.54 \text{ x } 20 = 4708 \text{ kN}$$

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$$

$$= \frac{5^2+(5x15)+15^2}{3(5+15)} = 5.42 \text{ m}$$

$$Z = \bar{x} + (\frac{P}{W} \text{ x } \frac{h}{3})$$

$$= 5.42 + (\frac{1589.22}{4708} \text{ x } \frac{18}{3}) = 7.45 \text{ m}$$



Eccentricity,

$$e = z - \frac{b}{2}$$

$$= 7.45 - \frac{15}{2} = -0.05 \text{ m}$$

Max stress,

$$\sigma_{max} = \frac{W}{b}(1 + \frac{6e}{b})$$

$$= \frac{4708}{15}(1 + \frac{6(0.05)}{15}) = \mathbf{320.144\ kN/m^2\ at\ (A)}$$

Min stress,

$$\sigma_{min} = \frac{W}{b}(1 - \frac{6e}{b})$$

$$= \frac{4708}{15}(1 - \frac{6(0.05)}{15}) = \mathbf{307.59\ kN/m^2\ (B)}$$

Result:

i)Max. stress, σ_{max} = 320.144 kN/m^2(comp) at (A)

ii)Min .stress, σ_{min} = 307.59 kN/m^2 (comp) at (B)

Problem: 7

A trapezoidal masonry dam is 3m wide at top. It is 52m high with a vertical water face retains water to a depth of 51m. Calculate the necessary minimum base width of the dam to ensure no tension is developed at the base. Weight of masonry is 24 kN/m^3 and weight of water is 10kN/m^3.

Given data

Top width, a = 3m

Height of dam, H= 52m

Height of water h= 51m

Specific weight of masonry, γ_m = 24 kN/m^3

Specific weight of water, γ_w = 10 kN/m^3

To Find:

Minimum base width, b = ?

Solution

Consider 1m length of wall

Lateral water pressure,

$P = \frac{\gamma_w h^2}{2}$

$= \frac{10x15^2}{2} = 1125$ kN

Weight of dam,

$W = \frac{(a+b)}{2}$ x γ_m x H

$= \frac{(3+b)}{2}$ x 24 x 52 = 624 (3+b)

$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$

$= \frac{3^2+(3b)+b^2}{3(3+b)} = \frac{9+(3b)+b^2}{3(3+b)}$

$Z = \bar{x} + \frac{P}{W} x \frac{h}{3}$

$= \frac{9+(3b)+b^2}{3(3+b)} + \frac{1125}{624(3+b)} x \frac{51}{3}$

$= \frac{9+(3b)+b^2}{3(3+b)} + \frac{91.94}{3(3+b)}$

$Z = \frac{b^2+(3b)+91.94}{3(3+b)}$ --------(1)

For no tension conditio**n,**

$e = \frac{b}{6}$

$z = \frac{b}{2} + e$

$= \frac{b}{2} + \frac{b}{6} = \frac{3b+b}{6}$

$z = \frac{2b}{3}$ ---------------(2)

equating (1) and (2)

$\frac{2b}{3} = \frac{b^2+(3b)+91.94}{3(3+b)}$

i.e., 2b (3+b) = b^2+ 3b + 91.94

i.e., 6b + $2b^2$- b^2 – 3b – 91.94 = 0

i.e., b^2 + 3b -91.94 = 0

a = 1 , b = 3 , c = - 91.94

$b = \frac{-3 \pm \sqrt{3^2-4x1x(-91.94)}}{2x1}$

= 8.205 m (or) -11.205m

Adopt b = 8.205 m

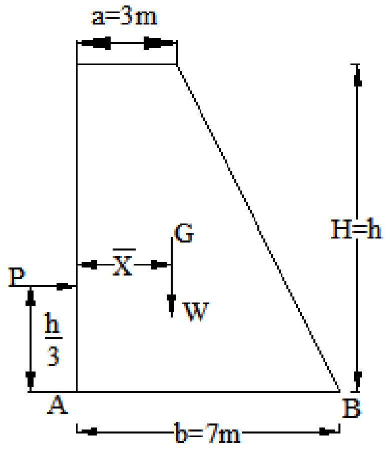
Result:

Min.base width of dam to ensure no tension = **8.205 m**

Problem:8

Calculate the minimum bottom width required for a dam of height 7m. Maximum depth of water to be impounded is 6m.The top width of section is 1.25 m. The specific weight of masonry is 22.5 kN/m^3. Take Co- efficient of friction between masonry and earth is 0.6.

Given data



Height of dam, $H = 7m$

Height of water, $h = 6m$

Top width, $a = 1.25m$

Specific weight of masonry, $\gamma_m = 22.5\ kN/m^3$

Take Specific weight of water, $\gamma_w = 9.81\ kN/m^3$

Co- efficient of friction between

Masonry and earth, $\mu = 0.6$

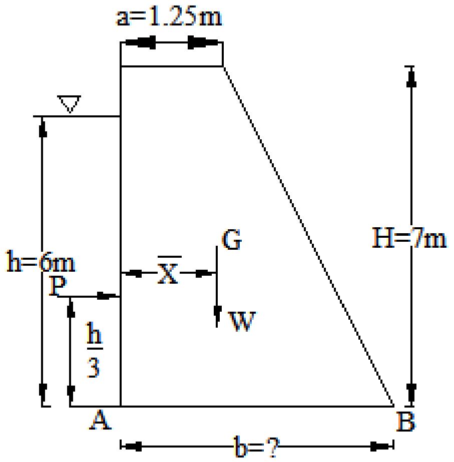
To find:

Minimum base width of the dam

i)To avoid tension at the base

ii)To avoid sliding

Solution



Consider 1m length of wall

Lateral water pressure,

$$P = \frac{\gamma_w . h^2}{2}$$

$$= \frac{9.81 x 6^2}{2} = 176.58kN$$

Weight of dam,

$$W = \frac{(a+b)}{2} \text{ x } \gamma_m \text{ x } H$$

$$= \frac{(1.25+b)}{2} \text{ x } 22.5 \text{ x } 7 = 78.75\ (1.25+b)\ kN$$

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$$

$$= \frac{1.25^2+(1.25b)+b^2}{3(1.25+b)} = \frac{1.563+(1.25b)+b^2}{3(1.25+b)}$$

$$Z = \bar{x} + \frac{P}{W} \text{ x } \frac{h}{3}$$

$$= \frac{1.563+(1.25b)+6^2}{3(1.25+b)} + \left(\frac{176.58}{78.75(1.25b)} \text{ x } \frac{6}{3}\right)$$

$$= \frac{1.563+(1.25b)+b^2}{3(1.25+b)} + \frac{13.454}{3(1.25+b)}$$

$$Z = \frac{b^2+(1.25b)+15.017}{3(1.25+b)} \quad \text{--------(1)}$$

(i) To avoid tension at the base

We know,

$$e = \frac{b}{6}$$

$$z = \frac{b}{2} + e$$

$$= \frac{b}{2} + \frac{b}{6} = \frac{2b}{3}$$

$$z = \frac{2b}{3} \quad \text{---------------(2)}$$

equating (1) and (2)

$$\frac{2b}{3} = \frac{b^2+(1.25b)+15.017}{3(1.25+b)}$$

i.e., 2b (1.25+b) = b^2+ 1.25b + 15.017

i.e., 2.5 b + $2b^2$- b^2 – 1.25b – 15.017 = 0

i.e., b^2 + 1.25b -15.017 = 0

a = 1 , b = 1.25 , c = - 15.017

$$b = \frac{-1.25 \pm \sqrt{1.25^2 - 4x1x(-15.017)}}{2x1}$$

= 3.3m (or) -4.55m

Adopt b = 3.3 m

(ii) To avoid sliding

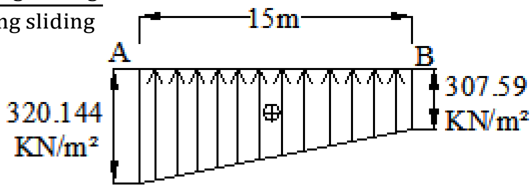
Force of safety against sliding = $\frac{\text{force resisting sliding}}{\text{force causing sliding}}$

i.e., $\frac{\mu.W}{\rho}$ = 1.5

$\frac{0.6 \text{ x } 78.75\ (1.25+b)}{176.58}$ = 1.5

i.e., 1.25 + b = 5.606

∴ b = 4.356m



Stress distribution diagram

Result: Minimum base width = max. of all values of b= 4.356 m

Problem: 10

A trapezoidal masonry dam 3m wide at the top and 7m wide at base has to retain water an its vertical face up to the top. Calculate maximum height of the dam to ensure no tension at the base. Take weight of masonry as 20 kN/m^3 and weight as 9.81 kN/m^3.

Given data

Top width, a = 3m

Bottom width, b = 7m

Height of dam H= height of water h

Specific weight of masonry, γ_m = 20 kN/m^3

Specific weight of water, γ_w = 9.81 kN/m^3

To Find:

Height of dam, H to avoid tension at the base

Solution

Consider 1m length of wall

Lateral water pressure,

$$P = \frac{\gamma_w . h^2}{2} \ (\therefore h = H)$$

$$= \frac{9.81 x H^2}{2} = 4.905\ H^2$$

Weight of dam,

$$W = \frac{(a+b)}{2} \text{ x } \gamma_m \text{ x } H$$

$$= \frac{(3+7)}{2} \text{ x 20 x H= 100 H}$$

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$$

$= \frac{3^2+(3x7)+7^2}{3(3+7)}$ = 2.633 m

$Z = \bar{x} + \frac{P}{W} x \frac{h}{3}$

$= 2.633 + \frac{4.905H^2}{100H} x \frac{H}{3}$

Z= 2.633 + 0.0164 H^2 -------------(1)

For no tension condition

$e = \frac{b}{6}$

$z = \frac{b}{2} + e$

$= \frac{b}{2} + \frac{b}{6} = \frac{2b}{3}$

$z = \frac{2x7}{3} = 4.667m$ ---------------(2)

equating (1) and (2)

2.633 + 0.0164 H^2 = 4.667

$\therefore H = 11.136\ m$

Result

Height of dam required to avoid tension, H = 11.136 m

REVIEW QUESTIONS

Two mark questions

1. What are the failure of Dams?
2. Define middle third rule.
3. State the shape of the elementary profile of a masonry dam.
4. State the conditions to avoid tension at the base.
5. Draw the elementary profile of a Masonry dam.
6. What is minimum base width of elementary profile of masonry dam?
7. Which are the main factors affecting the stability of a masonry dam?
8. When tension is developed at the base of a dam?
9. On what bass the base width of a masonry dam is determined?

Three mark questions

1. What are the causes of failure of masonry dams? State the conditions to check the stability of dams.
2. What is an elementary profile of a dam? Sketch the same.
3. A trapezoidal masonry dam having 12m height retains of water to a height of 10m on its vertical face. Find the horizontal water pressure if r =9.81KN/m^8.
4. State and explain middle third rule for no tension at the base of dam.
5. Derive the condition for no tension at the base of masonry dam.
6. State the procedure to find the minimum base width of a masonry dam for no tension.

Ten mark questions

1. A trapezoidal masonry dam m wide at top, 5m wide at the base is 8m high. It retains water to a depth of 7.5m on its vertical face. Calculate the maximum and minimum stress intensities at the base. Take weight of masonry as 22 kN/m^3 and that of water as 9.81 kN/m^3.
2. explain in details how the various checks are being done for ensuring the safety and stability of a gravity type masonry dam.

3. A trapezoidal masonry dam 1m wide at top,4m wide at its bottom and 6m high is retaining water on its vertical face to a height equal to the top of the dam. Determine the maximum and minimum intensities of stress at the base. Take weight of masonry as kN/m^3 and that of water as 9.81 kN/m^3.
4. A trapezoidal dam 3m wide at top, 8m wide at the base is 12 m high. Its retains water up to a depth of 11m on the upstream vertical face. Take the weight of masonry as24 kN/m^3 and that of water as 9.81 kN/m^3.Check the stability of the dam for overturning and sliding if μ= and F.O.S=1.5.
5. A trapezoidal masonry dam having 3m top width,8m bottom width and 12m high retains water to a height of 11m on its vertical face. Check the stability of the dam, if the masonry weighs 20 kN/m^3 and co-efficient of friction between the bottom of masonry and soil is 0.6.Take allowable compressive stress as 400 kN/m^3 and weight of water as 9.81 kN/m^3.
6. A trapezoidal masonry dam 2.5m wide at toip 5.5m wide at the base is 15m high. It retains water to a depth if 12m on its vertical face. Check the stability of the dam for overturning and sliding if μ=0.60 and F.O.S =1.5.Take weight of masonry as 25 kN/m^3 ands that of water as 9.81 kN/m^3.
7. A masonry dam of 11m height retains water on its vertical face for a height of 9m.The width of the dam is 2m in the top 2m height and varies gradually to 5m at bottom, with slope on one side only. Find the factor of safety of the dam against overturning if unit weights of masonry and water are 20 kN/m^3 and 10 N/m^3respectively.
8. A gravity dam of trapezoidal cross section of 20m height stores water on its vertical face for 18m height, with 2m free board. The top and bottom widths of dam are 4m and 10m respectively. Draw the pressure distribution diagram at base. specific weight of masonry ans water are 20 $kN/m^3$and 10 kN/m^3 respectively
9. A trapezoidal masonry dam 3m wide at top,12m wide at the base is 18m high. it retains water up to a depth of 17m on its vertical face. Check the stability of the dam for tension, sliding and crushing if μ=0.6 and F.O.S=1.5.Take the weight of masonry as 20 kN/m^3 and that of water as 10 kN/m^3.allowable compressive stress=400 kN/m^3.
10. A trapezoidal masonry dam 3m wide at top,7m wide at the base has retains water on its vertical face. Calculate the maximum height of the dam to ensure no tension at the base. take weight of masonry as 22 kN/m3 and that of water as 9.81 kN/m3.
11. A masonry retaining wall 1m wide at top, 3m wide at base retains earth on its its vertical face level with top. Determine the max height of dam required.

i) For no tension at base.

ii) To avoid sliding, F.S. against sliding=1.5

iii) To avoid over turning , F.S. against overturning=2.0

Take Unit weight of masonry = 23kN/m^3 and Unit weight of earth= 18 kN/m^3.

Angle of repose of soil=30°,μ=0.60.

12. A trapezoidal masonry dam 2m wide at top and 6m wide at base has to retain water on its vertical face up to the top. Calculate the maximum height of the dam to ensure that no tension is developed at the base. Take weight of masonry as 20kN/m^3 and weight of water as 9.81kN/m^3

5. DAMS AND RETAINING WALL

5.2 EARTH PRESSURE AND RETAINING WALLS

Definition – Angle of repose /Angle of Internal friction of soil– State of equilibrium of soil – Active and Passive earth pressures – Rankine's theory of earth pressure – Assumptions – Lateral earth pressure with level back fill / level surcharge (Angular Surcharge not required)– Earth pressure due to Submerged soils – (Soil retained on vertical back of wall only) – Maximum and minimum stresses at base **of** Trapezoidal Gravity walls – Stress distribution diagrams – Problems – Stability of earth retaining walls – Problems to check the stability of walls- Minimum base width for no tension.

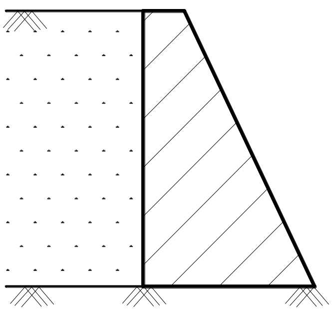
5.2. EARTH PRESSURE AND RETAINING WALL

5.2.1. Definition

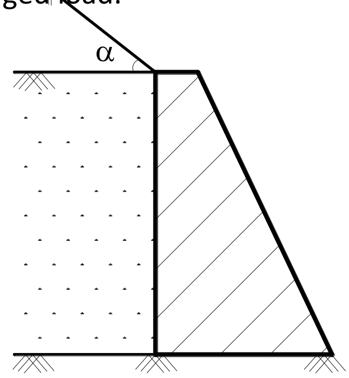
A masonry structure constructed to retain the earth is called retaining wall. The retained earth exerts pressure on the retaining wall is called earth pressure.

Types of retaining wall

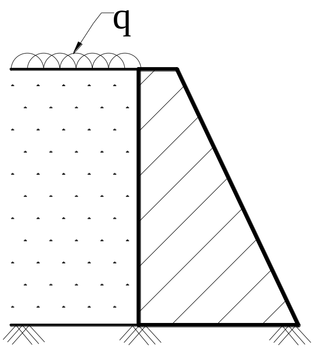
a. Based on the cross section
   1. Rectangular section
   2. Trapezoidal section with vertical (or) inclined back.
b. Based on the forms of back fill
   1. Retaining wall, earth level with top.
   2. Retaining wall, retaining earth surcharged soil.
   3. Retaining wall, retaining earth with surcharged load.
   4. Retaining wall retaining submerged soil.



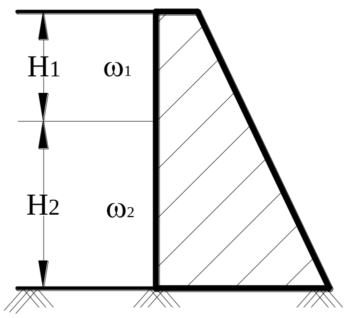
1) Retaining earth level with top



2) Retaining earth with surcharged soil



3) Retaining wall with surcharged load



4) Retaining earth with submerged soil

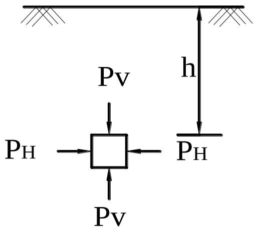
5.2.2. Angle of repose of soil (ϕ)

Earth cannot be retained at a steeper slope, as it tends to slide and slip. The maximum natural slope at which the soil particles will rest permanently due to internal friction, without further slipping (or) sliding is called angle of repose of soil. It is denoted by ϕ.

5.2.3. State of equilibrium of soil

1. Elastic equilibrium of soil

A soil mass in the natural state of rest is said to be in the state of elastic equilibrium.



Let

p_v = Vertical intensity of pressure.

$p_v = \omega h$

p_H= Intensity of lateral earth pressure.

$p_H = K_a \times P_v = K_a \times \omega h$

Where

K_a= Coefficient of earth pressure

ω = Specific weight of soil

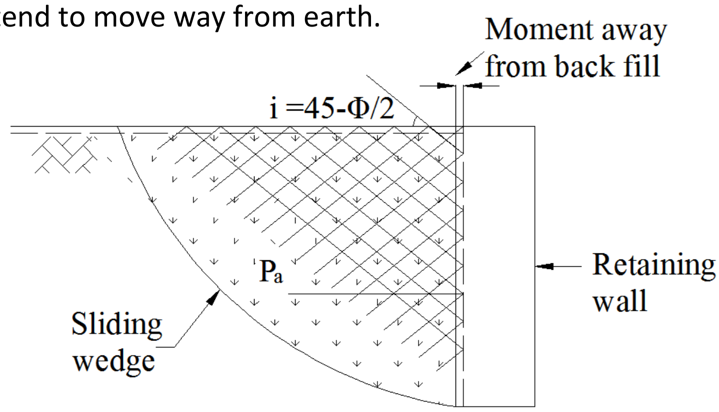
h = Depth of soil mass from free surface of earth.

2. Plastic equilibrium of soil

When the soil mass is allowed to retained or contract laterally, until failure takes place, the soil is said to be in plastic equilibrium of soil. The failure will be shear failure and exert the pressure may active earth pressure or passive earth pressure.

5.2.4.1 Active earth pressure

The pressure exerted by the retained earth on the retaining wall is called active earth pressure. Due to this pressure retaining wall tend to move way from earth.



p_a = Intensity of active earth pressure

$p_a = K_a \times \omega \times H$

Where

$$K_a = \left(\frac{1-\sin\phi}{1+\sin\phi}\right) = \tan^2(45 - \phi/2)$$

= coefficient of active earth pressure

5.2.4.1 Passive earth pressure.

The pressure exerted by the retaining wall (or) contract soil on the retained earth is called passive earth pressure. Due to this pressure retaining wall tend to move towards the earth. But it will happen rarely.

P_p = The intensity of passive earth pressure

$P_p = K_p \times \omega \times H$

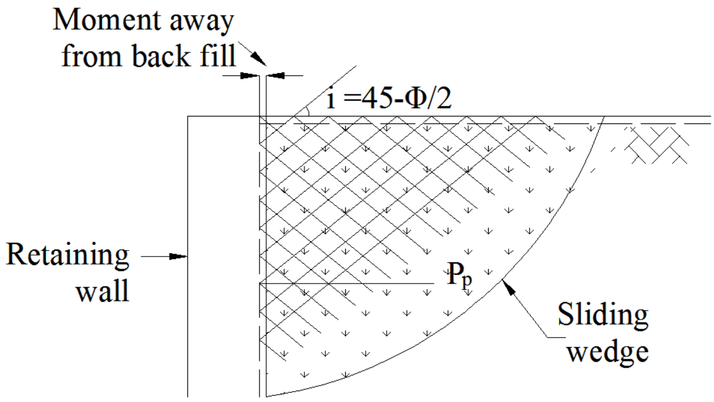
Where

$$K_p = \left(\frac{1+\sin\phi}{1-\sin\phi}\right) = \tan^2(45 + \phi/2)$$

= Coefficient of passive earth pressure

ω = Unit weight of earth

H = Height of retaining walls



5.2.5. Rankine's theory of earth pressure.

Rankine's theory and coulomb's theory of earth are available to determine the earth pressure on retaining walls.

British Engineer Prof. W.J. Rankine was given the theory of earth pressure in 1857.

5.2.6. Assumption made in theory of Rankine's earth pressure.

1. The retained soil mass is in the state of plastic equilibrium.
2. The retained soil mass is homogeneous, cohessionless.
3. The back of wall is smooth so that the frictional resistance between the wall and retained earth is negligible.
4. The retained soil surface is a straight line.
5. The failure of retained earth is by shear along a plane called rupture plane.

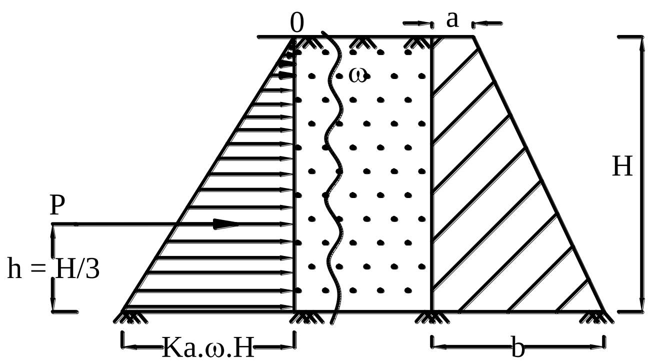
5.2.7. Rankine's lateral earth pressure on retaining wall

Case 1:

Retaining wall back fill level with top.

P = Rankine's lateral earth pressure

$$P = K_a \frac{\omega H^2}{2}$$



i) Pressure Diagram **ii) Section**

Where

$$K_a = \left(\frac{1-\sin\phi}{1+\sin\phi}\right) = \tan^2 (45 - \phi/2)$$

= coefficient of active earth pressure

ω = specific weight of earth

H = Height of retaining wall

ϕ = Angle of repose of soil

Total earth pressure P = Area of pressure diagram

$$P = \tfrac{1}{2} \times (K_a.\omega.H) \times H = \frac{K_a\,\omega H^2}{2}$$

$$P = K_a\, \omega H^2 / 2$$

This pressure will act at centre of gravity of pressure diagram.

ie at $\bar{h}$ from base

$$\bar{h} = H/3$$

Case 2:

Retaining wall back fill with surcharged soil

α = Angle of surcharge of soil

ϕ = Angle of repose of soil

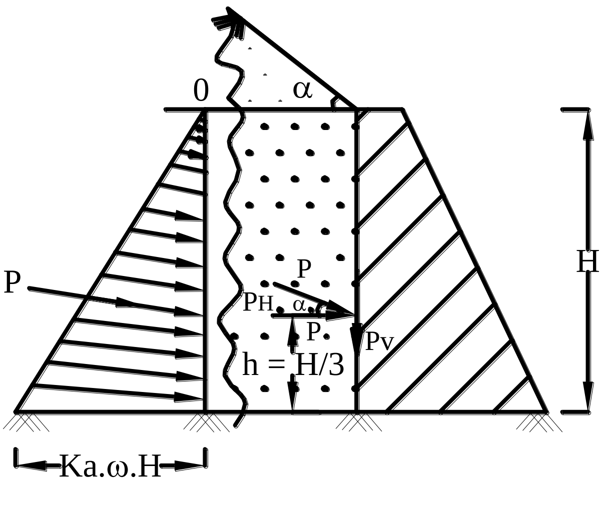
ω = Unit weight of earth

H = Height of retaining wall

$$K_a = \mathrm{Cos}_{\alpha}\ \frac{Cos\,\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\phi}}$$

= coefficient of active earth pressure

The lateral pressure will be parallel to the inclined earth surface.



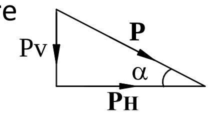
i) Pressure Diagram **ii) Section**

Rankines earth pressure $P = K_a \dfrac{\omega H^2}{2}$

This pressure will act parallel to the inclined earth surface.

P_H = Horizontal component of earth pressure

P_H = P x cos α



This will act at $\bar{h} = H/3$ from base.

P_V = Vertical component of earth pressure

P_V = P x Sinα

This will act along the vertical face of the wall.

Case 3:

Retaining wall with surcharged load

Let

q = Intensity of uniform surcharged load (or) superimposed load over the retained earth.

P = Total lateral earth pressure

P = $P_1 + P_2$

P_1 = Pressure due to surcharged load

P_1 = Area of pressure diagram of rectangle BCDE

P_1 = $(K_a \times q) \times H$

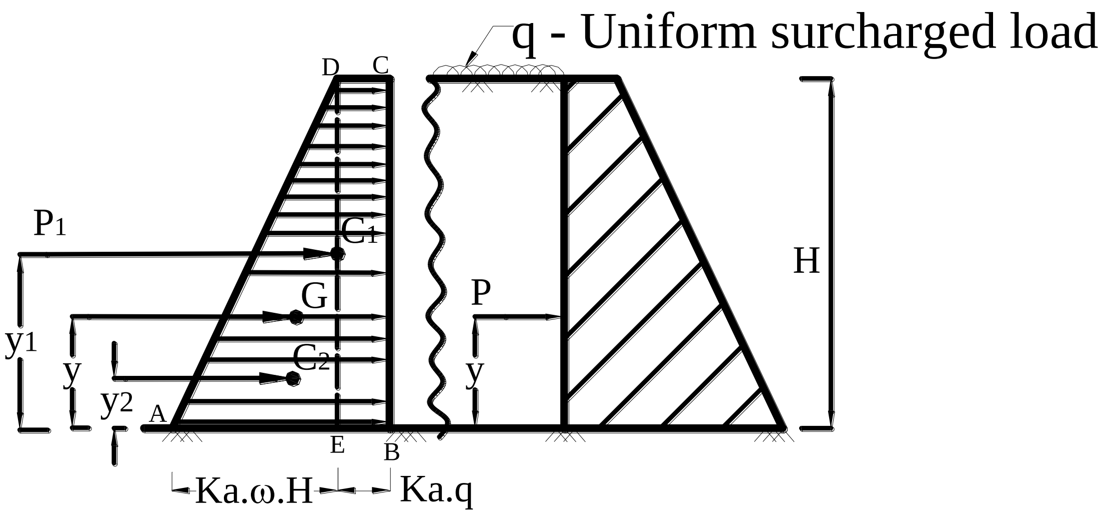
This will act at 'y_1' distance from base.

$$\boxed{y_1 = H/2}$$

P_2 = Pressure due to retained earth

P_2 = Area of pressure diagram of triangle ADE

P_2 = $\frac{1}{2} \times (K_a \times \omega.H) \times H = K_a \frac{\omega H^2}{2}$



Pressure Diagram **Section**

This pressure will act at y_2 distance from base

$y_2 = H/3$ distance from base.

P = $(P_1 + P_2)$ = resultant pressure

This will act at $\bar{y}$ distance from base.

Taking moment about bottom of wall.

$P \times \bar{y} = P_1 \times y_1 + P_2\, y_2$

$\bar{y} = \frac{1}{P} (P_1\, y_1 + P_2\, y_2)$

Case 4:

Retaining wall with sub merged soil.

When a part or entire depth of retained earth is submerged, the lateral pressure on the retaining wall is due to (i) Hydro static pressure (ii) Pressure due to dry earth (iii) Pressure due to the submerged weight of soil.

Let

ω_1 = Specific weight of dry soil to a depth H_1 from top.

$\omega 2$ = Specific weight of saturated soil to a depth H_2 from bottom.

ω = 9.81 = Specific weight of water kN/m^3.

$\omega_{(sub)}$ = $(\omega_2 - \omega)$ = Specific weight of submerged soil (or buoyant weight)

K_a = $\left(\frac{1-\sin\phi}{1+\sin\phi}\right)$ = coefficient of active earth pressure

P_1 = Pressure due to top soil to a depth H_1

P_1 = Area of pressure diagram of Δ^{le} section

P_1 = $K_a\,\omega_1\,H_1^2$

This will act at $y_1$distance from base $y_1 = \left(H_2 + {}^{H_1}\!/_3\right)$

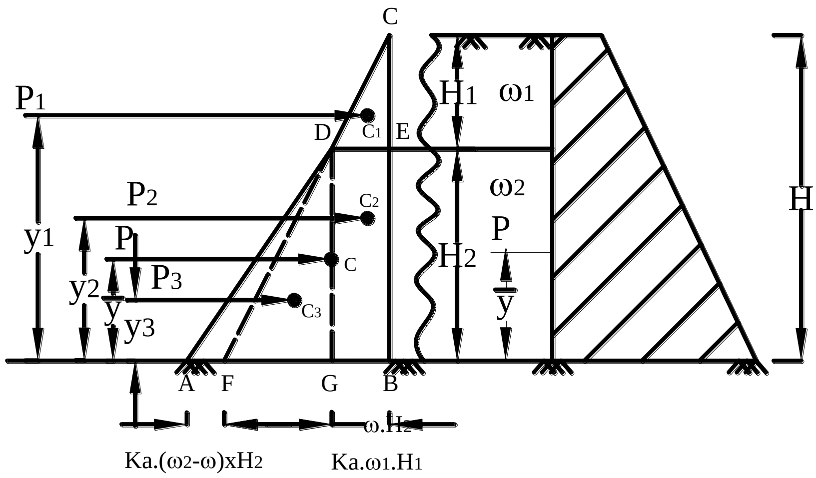
P_2 = Pressure due to top soil to a depth H_2

P_2 = Area of pressure diagram of rectangular section

P_2 = $(K_a\,\omega_1 \times H_1)\,H_2$

This will act at 'y_2' distance from bottom. $y_2 = {}^{H_2}\!/_2$

P_3 = Pressure due to hydrostatic fore (water) and submerged soil.



Pressure Diagram **Section**

P_3 = $\frac{1}{2}$ x $(\omega_{water}\,H_2 + K_a\,\omega_{sub}\,H_2)\,H$

This pressure will act at 'y_3' distance from base.

P = $P_1 + P_2 + P_3$ = (Total earth pressure)

This will act at $\bar{y}$ distance from base.

Taking moment about base

P x $\bar{y}$ = $(P_1.y_1 + P_2.y_2 + P_3.y_3)$

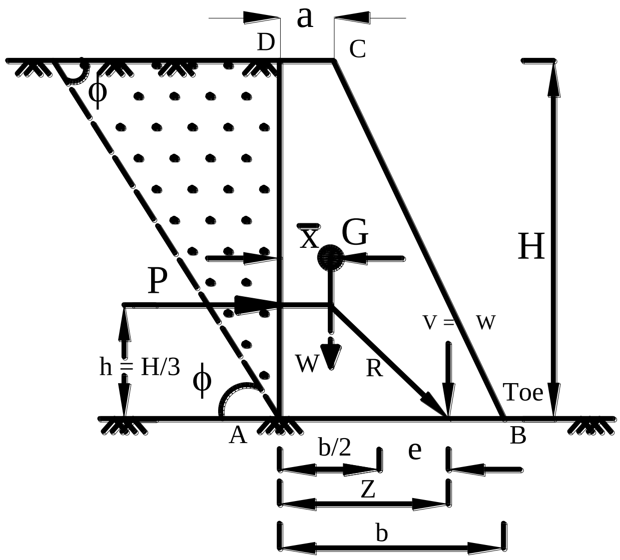
$\bar{y} = \frac{1}{P}(P_1 y_1 + P_2 y_2 + p_3 y_3)$

5.2.8. Maximum and Minimum stress

Consider a trapezoidal section masonry retaining wall, retaining earth on its vertical face level with top as shown in fig.

Let

a	= Top width of wall
b	= Bottom width of wall
H	= Height of retaining wall
γ	= Unit weight of masonry
ω	= Unit weight of soil
ϕ	= Angle of repose of soil



i. Weight of wall/m run (W)

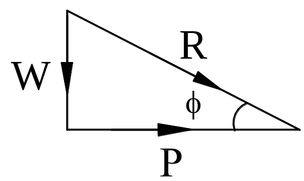
$W = \gamma$ volume $= \gamma \times \left(\frac{a+b}{2}\right) \times H \times 1$

$$\boxed{W = \gamma \times \left(\frac{a+b}{2}\right) \times H} \quad \text{kN}$$

This will act at $\bar{x}$ from vertical face.

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$$

Rankine's earth pressure/m (P)



$P = K_a \frac{\omega H^2}{2}$

Where

$K_a = \left(\frac{1-\sin\phi}{1+\sin\phi}\right) = \tan^2(45 - \phi/2)$

= coefficient of active earth pressure

This will act at $\bar{h}$ from base $\bar{h} = H/3$

V = Total vertical force at the base

V = W.

R = Resultant thrust [∴ V = W]

$R = \sqrt{V^2+P^2} = \sqrt{W^2+P^2}$

Position of resultant thrust (Z)

Taking moment about Heel (A)

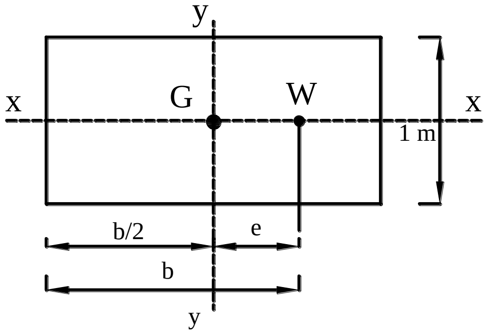
$W \times Z = W.\ \bar{x} + P \times \bar{h}$

$$\boxed{Z = \bar{x} + \left(\frac{P}{W}\right)\bar{h}}$$

$e = \left(Z - \frac{b}{2}\right)$ = eccentricity of resultant thrust

$M = (W \times e)$ = moment due to eccentric force

Area $A = b \times 1$



$$Z_y = \frac{I_{yy}}{\bar{x}} = \frac{\left(\frac{db^3}{12}\right)}{\left(\frac{b}{2}\right)} = \frac{\left(1 x \frac{b^3}{2}\right)}{\frac{b}{2}}$$

$$Z_y = \frac{b^2}{6}$$

Stresses at base (σ)

$\sigma_c = \frac{W}{A}$ = Direct compressive stress due to weight of wall

$\sigma_b = \frac{M}{Z} = \frac{W \, x e}{\frac{b^2}{6}} = \frac{6W.e}{b^2}$ = Bending

$\sigma = \sigma_c \pm \sigma_b$ = Total stress at base

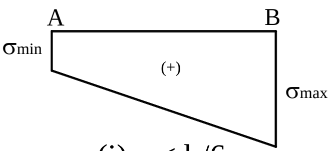
$$\sigma = \left(\frac{W}{(b\,x1)} \pm \frac{6W.e}{b^2}\right) = \frac{W}{b}\left(1 \pm \frac{6e}{b}\right)$$

$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right)$ = maximum stress at toe (B)

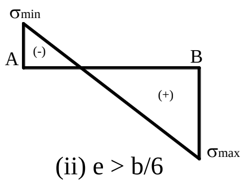
$\sigma_{min} = \frac{W}{b}\left(1 - \frac{6e}{b}\right)$ = minimum stress at heel (A)

Stress distribution diagram

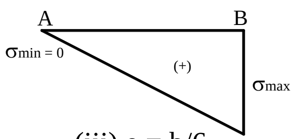
The value of maximum and minimum stresses are based on the eccentricity (e) of vertical force at base as given below.



(i) e < b/6



(ii) e > b/6



(iii) e = b/6

5.2.9. Stability of retaining wall

Causes of failures of retaining wall.

1. Tension at the base of wall
2. Sliding of wall along the base
3. Overturning of wall about toe.
4. Crushing of masonry at the base of wall.

Stability of retaining wall

Following are the conditions of stability of wall.

i. To avoid tension at base

$e < \frac{b}{6}$ (or) $Z \leq \frac{2}{3}$ b

$\sigma_{min} \geq \left(1 - \frac{6e}{b}\right) \geq 0$

ii. To avoid sliding

F.S. (Sliding) = $\frac{Total\ frictional\ force}{Horizontal\ force} \geq 1.0$

F.S. (Sliding) = $\frac{\mu\, x\, w}{P} \geq 1.0$

for design purpose F.S. ≥ 1.5

iii. To avoid overturning

F.S. (Overturning) = $\frac{Balancing\ moment}{Overturning\ moment} \geq 1.0$

F.S. (Overturning) = $\frac{W\,(b-\bar{x})}{P\, x\, \bar{h}} \geq 1.0$

Design purpose FS = 1.5 to 2.00

iv. To avoid crushing

Maximum compressive stress should be less than allowable compressive stress. (Safe bearing capacity of soil)

σ_{max} < SBC of soil.

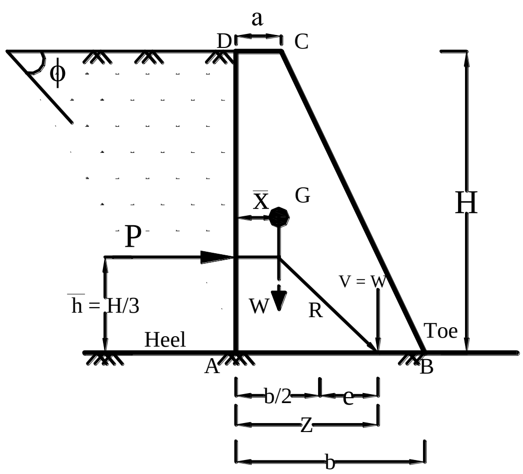
RETAINING WALL

Problem 1

A trapezoidal masonry retaining wall 1m wide at top, 3m wide at its bottom is 8m high. It retaining earth having level with the top of the wall on its vertical face. Find the max. min stress intensities at the base of the wall. If wt. of masonry is 24 kN/m^3 and earth is 18 kN/m^3 the angle of repose of earth is 40°.

Given

a = 1m
b = 3m
H = 8m
γ = 24 kN/m^3
ω = 18 kN/m^3
ϕ = 40°



Required

$\sigma = \frac{w}{b}\left(1 \pm \frac{6e}{b}\right) = ?$

Solution

i. Weight of wall per metre (W)

W $= \gamma \left(\frac{a+b}{2}\right) x H$ $= 24 \left(\frac{1+3}{2}\right) x 8$ = $\boxed{W = 384\ kN}$

This will act at $\bar{x}$ distance from vertical face

$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} = \frac{1^2 + (1x3) + 3^2}{3(1+3)} = \boxed{x = 1.08m}$

ii. Rankine's earth pressure per metre run (P)

$$P = Ka.\left(\frac{\omega H^2}{2}\right)$$

$$Ka = \left(\frac{1-\sin\phi}{1+\sin\phi}\right) = \left(\frac{1-\sin 40°}{1+\sin 40°}\right) = 0.2174$$

$$P = 0.2174\, x\, \frac{18\, x\, 8^2}{2}$$

$$\boxed{P = 125.25\ \text{kN}}$$

This will act at a distance $\bar{h}$ from base

$$\bar{h} = H/3 = 8/3$$

$$\boxed{\bar{h} = 2.67m}$$

iii. Position of resultant thrust (R)

$$Z = \bar{x} + \frac{P}{W}\bar{h}$$

$$Z = 1.08 + \left[\frac{125.25}{384}\, x\, 2.67\right] = \boxed{Z = 1.95\text{m}}$$

Eccentricity (e)

$$e = Z - b/2 = (1.95 - 3/2) = \boxed{e = 0.45}$$

Stress at base (σ)

$$\sigma = \frac{W}{b}\left(1 \pm \frac{6e}{b}\right)$$

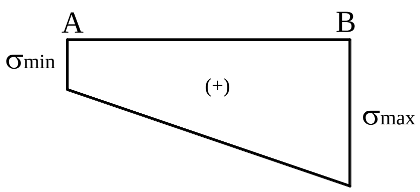
$$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right) \text{ at toe } = \frac{384}{3}\left(1 + \frac{6\, x\, 0.45}{3}\right)$$

$$\boxed{\sigma_{max} = 243.2\ \text{kN/m}^2}$$ (Compression)

$$\sigma_{min} = \frac{w}{b}\left(1 - \frac{6e}{b}\right) = \frac{384}{3}\left(1 - \frac{6\, x\, 0.45}{3}\right) \text{ (Compression)}$$

$$\boxed{\sigma_{min} = 12.8\ \text{kN/m}^2}$$

Stress diagram



Problem 2

A retaining wall 1m wide at top 3m wide at base 6m high retains earth on its vertical face leveled with top unit wt. of masonry and earth are 23 kN/m^3 and 18 kN/m^3 respectively. Determine

- **i. Rankines earth pressure 1m run**
- **ii. Resultant thrust**
- **iii. Stresses at the base**
- **iv. Check the stability of wall**

Take μ = 0.6, max allowable stress = 300 kN/m^2 the angle of repose of soil is 30°.

Given

a = 1m

b = 3m

H = 6m

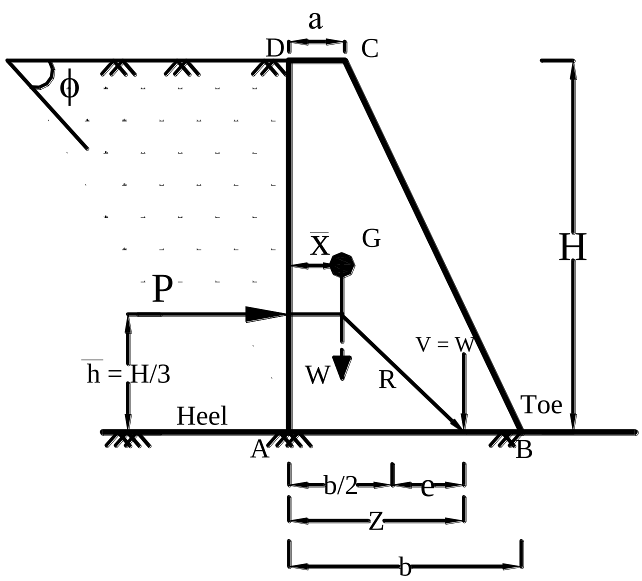
γ = 23 kN/m^2

ω = 18 kN/m^2

ϕ = 30°

μ = 0.6

SBC = 300 kN/m^2



Required

$$\sigma = \frac{w}{b}\left(1 \pm \frac{6e}{b}\right)$$

Solution

i. Weight of wall 1m (w)

$$W = \gamma\left(\frac{a-b}{2}\right)H = 23x\left(\frac{1+3}{2}\right)x6$$

$$\boxed{W = 276 \text{ kN}}$$

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} = \frac{1^2+(1x3)+3^2}{3(1+3)}$$

$$\boxed{x = 1.08\text{m}}$$

ii. Rankine's earth pressure per metre (P)

$$P = Ka.\left(\frac{\omega H^2}{2}\right)$$

$$Ka = \left(\frac{1-\sin\phi}{1+\sin\phi}\right) = \left(\frac{1-\sin 30°}{1+\sin 30°}\right) = 0.333$$

$$P = 0.333 \text{ x } \frac{18 x 6^2}{2}$$

$$\boxed{P = 108 \text{ kN}}$$

This will act at a distance $\bar{h}$ from base

$$\bar{h} = H/3 = 6/3$$

$$\boxed{\bar{h} = 2m}$$

Resultant earth pressure (R)

R = $\sqrt{W^2+P^2}$ = $\sqrt{276^2+108^2}$ $\boxed{R = 296.38\text{ kN}}$

Position of Resultant pressure (Z)

Z = $\bar{x}+\frac{P}{W}\bar{h}$ = 1.08 + $\frac{108}{276}$ x 2 $\boxed{Z = 1.86\text{m}}$

Eccentricity (e)

e = Z – $b/2$ = 1.86 – $3/2$ $\boxed{e = 0.36\text{m}}$

Stress at base

σ = $\frac{W}{b}\left(1\pm\frac{6e}{b}\right)$

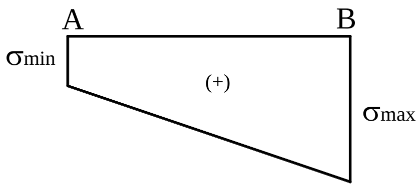
σ_{max} = $\frac{W}{b}\left(1+\frac{6e}{b}\right)$ = $\frac{276}{3}\left(1+\frac{6\,x\,0.36}{3}\right)$ at toe

$\boxed{\sigma_{max} = 158.24\text{ kN/m}^2}$ (Compression)

σ_{min} = $\frac{w}{b}\left(1-\frac{6e}{b}\right)$ = $\frac{276}{3}\left(1-\frac{6\,x\,0.36}{3}\right)$ at heel

$\boxed{\sigma_{min} = 25.76\text{ kN/m}^2}$ (Compression)

Stress diagram



Check the stability of wall

i. To avoid tension

σ_{min} ≥ 0 (or) e $\leq b/6$ (or) z $\leq 2/3$ b

e = 0.36 m

$b/6$ = $3/6$ = 0.5m

e < $b/6$ (Safe)

Hence tension will not be developed at base

iii. To avoid sliding

F.S. = $\frac{\mu\, w}{P} \geq 1.0$

F.S. (Sliding) = $\frac{0.6\,x\,276}{108}$ = 1.53

1.53 $\geq$ 1.0 ; **Hence safe against sliding.**

iii. To avoid failure from crushing

SBC > σ_{max}

SBC = 300 kN/m^2

σ_{max} = 158.24 kN/m^2

SBC > σ_{max} (Hence safe)

iv. To avoid over turning

$$F.S. = \frac{W\,(b-\bar{x})}{P\,\bar{h}} = 1.0$$

$$F.S. = \frac{276(3-1.08)}{108\,x\,2} \qquad = 2.45 > 1.0 \text{ (Hence Safe)}$$

Hence safe against over turning.

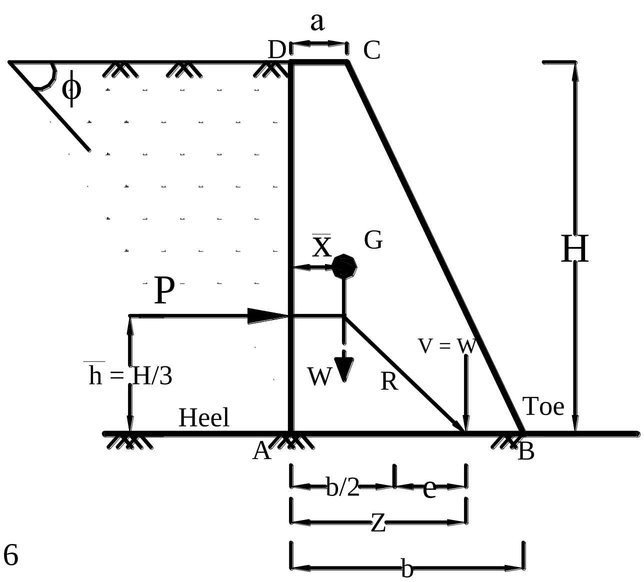
Problem 3

A retaining wall 1m wide at top 6m high retains earth on its vertical face level with top. The unit wt. of masonry and earth are 23 kN/m^3 and 16 kN/m^3 respectively. Determine the min. base width required to avoid.

i) Tension and ii) Sliding F.S. against sliding = 1.5

The angle of repose of soil is 30°, Take μ = 0.6

Given



a = 1m

H = 6m

μ = 0.6

γ = 23 kN/m^3

ω = 16 kN/m^3

ϕ = 30°

Required

b = ?

Solution

i. Weight of wall per metre (W)

$$W = \gamma\left(\frac{a+b}{2}\right)H \quad = 23\left(\frac{1+b}{2}\right)x\,6$$

$$\boxed{w = 69\ (1 + b)\ \text{kN}} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} \qquad = \frac{1^2+1x\,b+b^2}{3(1+b)}$$

$$\boxed{\bar{x} = \frac{1+b+b^2}{3(1+b)}} \qquad \ldots\ldots\ldots\ldots \quad (2)$$

ii. Rankine's earth pressure (P)

$$P = Ka.\left(\frac{\omega H^2}{2}\right)$$

$$Ka = \left(\frac{1-\sin\phi}{1+\sin\phi}\right) = \left(\frac{1-\sin 30°}{1+\sin 30°}\right) = 0.333$$

$$P = 0.333 \text{ x } \frac{16\,x\,6^2}{2}$$

$$\boxed{P = 96\ \text{kN}}$$

iii. Position of Resultant pressure (Z)

$$Z = \bar{x} + \frac{P}{W}\bar{h} = \frac{1+b+b^2}{3(1+b)} + \frac{96}{69(1+b)} \times \frac{6}{3}$$

$$= \frac{1+b+b^2}{3(1+b)} + \frac{0.348}{3(1+b)} = \frac{b^2+b^1+1+8.348}{3(1+b)}$$

$$\boxed{Z = \frac{b^2+b^1+9.348}{3(1+b)}}$$

Maximum base width required (b)

a) To avoid tension

$Z \leq 2/3\ b$

$$\frac{2}{3}b = \frac{b^2+b+9.348}{3(1+b)}$$

$2b(1+b) = b^2 + b + 9.348$

$2b + 2b^2 = b^2 + b + 9.348$

$b^2 + b - 9.348 = 0$

$$b = \frac{-1 \pm \sqrt{1^2 - 4 x 1 x (-9.348)}}{2}$$

$$\boxed{b_1 = 2.6m}$$

b) To avoid sliding

$$F.S = \frac{\mu w}{P} \geq 1.5$$

$$\frac{0.6 x 69(1+b)}{96} = 1.5$$

$$1 + b = \frac{96}{0.6 x 69} \times 1.5 = 3.48$$

$b = 3.48 - 1$

$$\boxed{b_2 = 2.48\ m}$$

Minimum base width = Maximum of b_1 & b_2

Result

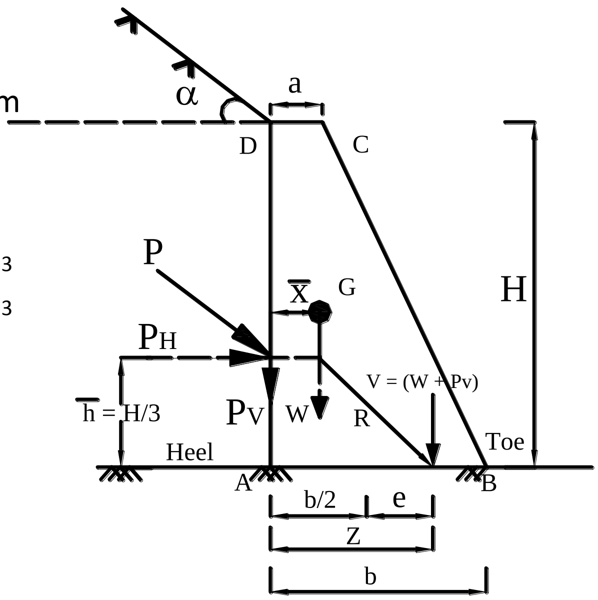
Minimum base width required (b) = 2.6m

Problem 4

A masonry retaining wall of trapezoidal section with a vertical face of 1m wide at top and 3m wide at bottom with a height of 6m. It retains sand over the entire height with an angle of surcharge 20°. Determine the stress intensities at the base of the wall. The sand wt. is 18kN/m^3 and an angle of repose of soil is 30°. The masonry wt. of 24 kN/m^3.

Given

Top width	a	= 1m
Bottom width	b	= 3m
Height	H	= 6m
Angle of surcharge	α	= 20°
Angle of repose of soil	ϕ	= 30°
Weight of masonry	γ	= 24 kN/m^3
Weight of soil	ω	= 18 kN/m^3



Required

$$\sigma = \frac{V}{b}\left(1 \pm \frac{6e}{b}\right)$$

Solution

i. Weight of wall run per metre run (W)

$$W = \gamma\left(\frac{a+b}{2}\right)H = 24\left(\frac{1+3}{2}\right)6$$

$$\boxed{W = 288 \text{ kN}}$$

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} = \frac{1^2+(1x3)+3^2}{3(1+3^b)}$$

$$\boxed{\bar{x} = 1.08\text{m}}$$

ii. Rankines earth pressure per metre run (P)

$$P = Ka \text{ x } \left(\frac{\omega H^2}{2}\right)$$

$$Ka = \left(\cos\alpha \frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\phi}}\right)$$

$$Ka = \left(\cos 20° \frac{\cos 20° - \sqrt{\cos^2 20° - \cos^2 30°}}{\cos 20° + \sqrt{\cos^2 20 - \cos^2 30°}}\right)$$

$$Ka = 0.9397 \text{ x } \frac{0.9397 - \sqrt{0.9930 - 0.75}}{0.9397 + \sqrt{0.883 - 0.75}}$$

$$Ka = 0.9397 \text{ x } \left(\frac{0.9397 - 0.3647}{0.9397 + 0.3647}\right) = 0.9397 \text{ x } \frac{0.575}{1.3044}$$

$$Ka = 0.414$$

$$P = Ka \text{ x } \omega \; H^2/2 = 0.414 \text{ x } 18 \text{ x } 6^2/2 = 134.14 \text{ kN}$$

P_V = Vertical component of earth pressure

P_V = P x sin α = 134.14 x sin 20° = 45.88 kN

$$\boxed{P_v = 45.88 \text{ kN}}$$

This will act along the vertical face of wall

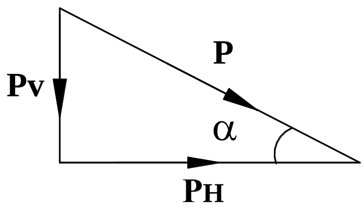
P_H = Horizontal component of earth pressure

P_H = P x cos α

This will act at $\bar{h}$ distance from base.

$\bar{h}$ = $H/3$ = $6/3$ = 2m

$\boxed{\bar{h} = 2\text{m}}$



Total vertical force (V)

V = W + P_V = (288 + 45.88) = 333.88 kN

V = 333.88 kN

Resultant thrust per metre run (R)

R = $\sqrt{V^2 + {P_H}^2} = \sqrt{333.88^2 + 126^2}$

R = 356.86 kN

This resultant pressure will hits the base at 'Z' distance from vertical face.

Position of resultant thrust (Z)

Taking moment about A (heel)

V x Z = (W $\bar{x}$) + (P_H x $\bar{h}$) + (P_V x 0)

Z = $\frac{1}{V}$ (W $\bar{x}$ + P_H $\bar{h}$)

Z = $\frac{1}{333.88}$ ((288 x 1.08) + (126 x 3))

Z = 1.60m

Eccentricity e = (Z – $b/2$) = $\left(1.60 - \frac{3}{2}\right)$ = 0.1m

Stress intensities at the base (σ)

σ = $\frac{V}{b}\left(1 \pm \frac{6e}{b}\right)$

σ_{max} = $\frac{V}{b}\left(1 \pm \frac{6e}{b}\right)$ at toe (B)

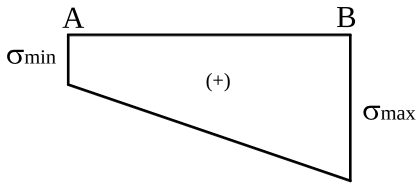
σ_{max} = $\frac{333.88}{3}\left(1 + \frac{6\,x\,0.1}{3}\right)$ = 133.55 kN/m^2 (Comp.)

σ_{min} = $\frac{V}{b}\left(1 \pm \frac{6e}{b}\right)$ at heel (A)

σ_{min} = $\frac{333.88}{3}\left(1 + \frac{6\,x\,0.1}{3}\right)$

σ_{min} = 89.03 kN/m^2 (Comp.)

Stress diagram



Problem 5

A retaining wall 1.5m wide at top 4m wide at base and 8m high retains earth on its vertical face with surcharge of 15° and angle of repose of 30° unit wt. of masonry and earth are 23 kN/m^3 and 18 kN/m^3 of the respectively. Determine

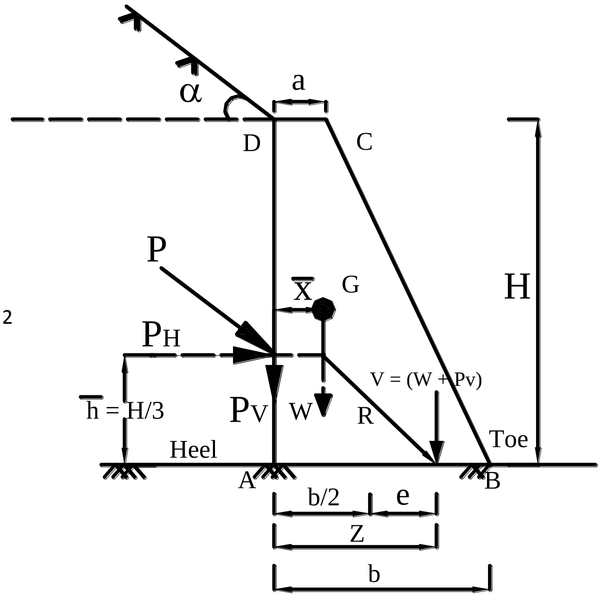
i. **Rankine's earth pressure**
ii. **Stress intensities at base**
iii. **Check the stability of wall**

for tension sliding, overturning and crushing,

Take μ = 0.6, maximum allowable stress = 300 kN/m^2

Given

Top width	a	= 1.5m
Bottom width	b	= 4m
Height	H	= 8m
Angle of repose of soil	ϕ	= 30°
Weight of masonry	γ	= 23 kN/m^3
Weight of soil	ω	= 18 kN/m^3
	μ	= 0.6
Maximum allowable stress		= 300 Kn/m^2
Angle of repose	α	= 15°



Required

i) $\sigma = \frac{V}{b}\left(1 \pm \frac{6e}{b}\right)$

ii) Check for the stability.

Solution

i. Weight of wall per metre run (W)

$$W = \gamma\left(\frac{a+b}{2}\right)H = 23\left(\frac{1.5+4}{2}\right)8$$

$$\boxed{W = 506 \text{ kN}}$$

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} = \frac{1.5^2+(1.5x4)+4^2}{3(1.5+4)}$$

$$\boxed{\bar{x} = \mathbf{1.47m}}$$

ii. Rankine's earth pressure per metre run (P)

$$Ka = (\cos\alpha)\left(\frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2 - \cos^2\phi}}\right)$$

Ka = (cos 15°) $\left(\dfrac{\cos 15° - \sqrt{\cos^2 15° - \cos^2 30°}}{\cos 15° + \sqrt{\cos^2 15 - \cos^2 30}}\right)$

Ka = 0.9659 x $\dfrac{0.9659 - \sqrt{0.933 - 0.75}}{0.9659 + \sqrt{0.933 - 0.75}}$

Ka = 0.9659 x $\left(\dfrac{0.9659 - 0.4277}{0.9659 + 0.4277}\right) = 0.9659\,x\,\dfrac{0.5382}{1.3936}$

Ka = 0.373

P = Ka x $\dfrac{\omega H^2}{2}$ = 0.373 x $\left(\dfrac{18 x 8^2}{2}\right)$

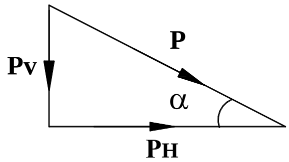
P = 214.86 kN

P_V = Vertical component of earth pressure

P_V = P x sin α

= 214.86 x sin 15°

$\boxed{P_V = 55.60 \text{ kN}}$



This will act along the vertical face of wall.

P_H = Horizontal component of earth pressure

P_H = P cos α

Pv = 214.86 x cos 15°

$\boxed{P_H = 207.54 \text{ kN}}$

This will act as $\bar{h}$ from base.

$\bar{h}$ = $H/3$ = $8/3$ = 2.67

$\boxed{\bar{h} = 2.67}$

Total vertical force (V)

V = W + P_V = 506 + 55.60

$\boxed{V = 561.60 \text{ kN}}$

Resultant thrust (R)

R = $\sqrt{V^2 + P_H{}^2}$ = $\sqrt{561.60^2 + P_H{}^2}$

$\boxed{R = 598.72 \text{ kN}}$

Position of resultant thrust (Z)

Taking moment about A.

Z = $\dfrac{1}{V}$ [W $\bar{x}$ + (P_H $\bar{h}$) + (P_V x 0)] = $\dfrac{1}{V}$ [W $\bar{x}$ P_H $\bar{h}$ + (P_V x 0)]

Z = $\dfrac{1}{561.60}$ (560 x 1.47 + 207.54 x 2.67)

$\boxed{Z = 2.31\text{m}}$

Eccentricity (e)

e $= Z - b/2 = 2.45 - 4/2$

$\boxed{\mathbf{e = 0.45m}}$

Stresses at base (σ)

$$\sigma = \frac{V}{b}\left(1 \pm \frac{6e}{b}\right)$$

$$\sigma_{max} = \frac{V}{b}\left(1 + \frac{6e}{b}\right) \text{ at toe (B)}$$

$$\sigma_{max} = \frac{561.60}{4}\left(1 + \frac{6 \times 0.45}{4}\right) \text{ at toe}$$

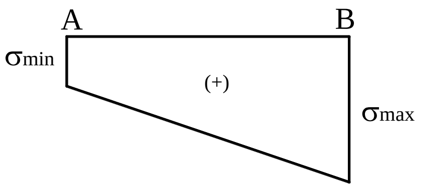
$$\boldsymbol{\sigma_{max} = 235.08\ kN/m^2\ (Comp.)}$$

$$\sigma_{min} = \frac{V}{b}\left(1 + \frac{6e}{b}\right) \text{ at heel (A)}$$

$$= \frac{561.60}{4}\left(1 - \frac{6 \times 0.45}{4}\right)$$

$$\boldsymbol{\sigma_{min} = 45.63\ kN/m^2\ (Comp.)}$$

Stress diagram



Check the stability of wall

(i) To avoid tension at base

$\sigma_{min} \geq 0$

$e \leq b/6$ (or) $Z \leq 2/3\ b$

$e = 0.45$ m

$b/6 = 4/6 = 0.67$ m

$e < b/6$

Hence safe against tension.

(ii) F.S. against sliding

$$\text{F.S.} = \frac{\mu V}{P_H} \geq 1.0$$

$$= \frac{0.6 \times 561.60}{207.54}$$

$= 1.62 > 1.0$ (Safe)

Hence safe against sliding.

(iii) F.S. against overturning

F.S. $= \dfrac{V(b-\bar{x})}{P_H \bar{h}} = \dfrac{561.60(4-1.47)}{207.54 x 2.67}$

F.S = 2.56 > 1.0 Safe.

Hence safe against overturning.

(iv) To avoid crushing

SBC > σ_{max}

360 > 235.08 kN/m^2 (Safe)

Hence safe against crushing.

Problem 6

A trapezoidal section retaining wall 2m wide at top, 4m wide at base and 6m high retains earth level with top. The retained earth which transmits a uniform surcharged load 60 kN/m^2. The angle of repose of soil is 30°. The unit weight of masonry and earth are 24 kN/m^3 and 18 kN/m^3 respectively, calculate the magnitude and position of line of action and position of line of action of horizontal earth pressure per meter length of wall.

Given

Top width	a	= 2m
Bottom width	b	= 4m
Height	H	= 6m
Uniform surcharged load q		= 60 kN/m^2
Angle of repose of soil ϕ		= 30°
Specific weight of masonry γ		= 24 kN/m^3
Specific weight of earth ω		= 18 kN/m^3

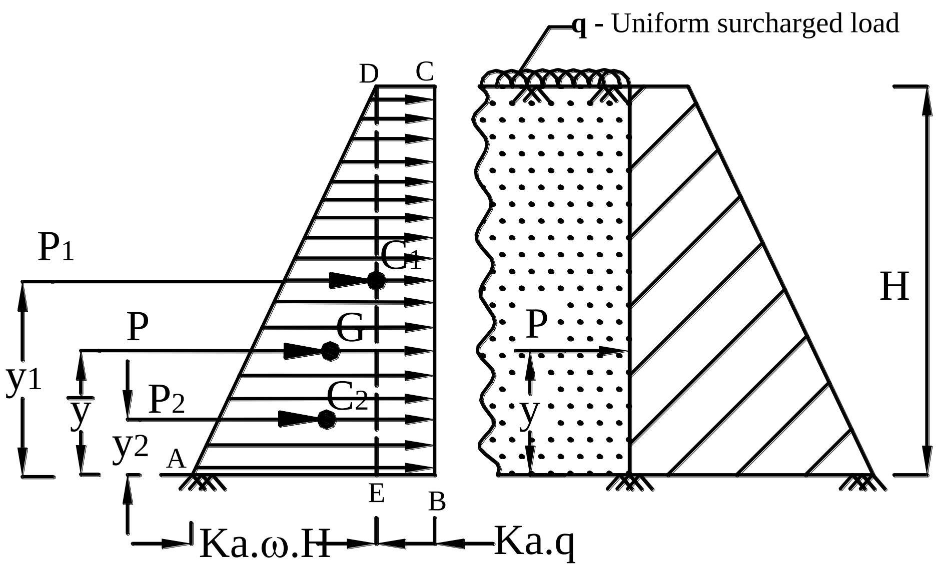
Required

i) Horizontal pressure P = ?

ii) Position of horizontal pressure $\bar{y}$ = ?

Solution

Draw pressure diagram as shown in fig.



i. Coefficient of active earth pressure (Ka)

Ka $= \frac{1-\sin\phi}{1+\sin\phi} = \left(\frac{1-\sin 30}{1+\sin 30}\right) = 0.333$

Horizontal earth pressure/m run (P)

P $= P_1 + P_2$

P_1 = Pressure due to surcharged load

P_1 = Area of rectangle BCDE

P_1 = (Ka x q) x H = 0.333 x 60 x 6 = 120 kN

This will act at y_1 distance from base.

$y_1 = \frac{H}{2} = \frac{6}{2} = 3\text{m}$

P_2 = Pressure due to retained earth

P_2 = Area of triangle AED

$P_2 = \frac{1}{2}$ x (Ka x ω x H) x H = $\frac{1}{2}$ (0.333 x 18 x 6) x 6

P_2 = 108 kN

This will act at y_2 distance from base.

$y_2 = \frac{H}{3} = \frac{6}{3} = 2\text{m}$

Total horizontal pressure P = $P_1 + P_2$ = (120 + 108) = 228 kN.

This pressure will act at $\bar{y}$ distance from bottom.

Position of horizontal earth pressure (from base $\bar{y}$)

Taking moment about bottom of wall.

P x $\bar{y} = P_1 y_1 + P_2 y_2$

$\bar{y} = \frac{1}{P}(P_1 y_1 + P_2 y_2) = \frac{1}{228}(120 \times 3 + 108 \times 2)$

$\bar{y}$ = 2.52m from base

Result

P = 228 kN

$\bar{y}$ = 2.52m

Problem 7

A trapezoidal retaining wall of 6m height is 2 metre wide at top 4 m wide at its bottom. It retains earth on its vertical face to its full height. The bottom layer of soil of 1.5m height is fully submerged in water, check the stability of the wall against overturning. If the unit weights of dry soil, wet soil and masonry are 16 kN/m^3, 19 kN/m^3 and 23 kN/m^3 respectively and angle of repose of soil is 30°. Take unit weight of water 10 kN/m^3. (Oct. 2009)

Given

Top width	a	= 2m
Bottom width	b	= 4m
Height	H	= 6m
	H_2	= 1.5m
	H_1	= (6 – 1.5) = 4.5m

ω_1 = 16 kN/m^3

ω_2 = 19 kN/m^3

Weight of masonry γ = 23 kN/m^3

$\omega_{(water)}$ = 10 kN/m^3

ϕ = 30°

Unit weight of submerged soil $\omega_{(sub)} = (\omega_2 - \omega_{water})$

$\omega_{(sub)}$ = (19 – 10) = 9 kN/m^3

Required

Check the stability

Solution

Consider one metre length of wall.

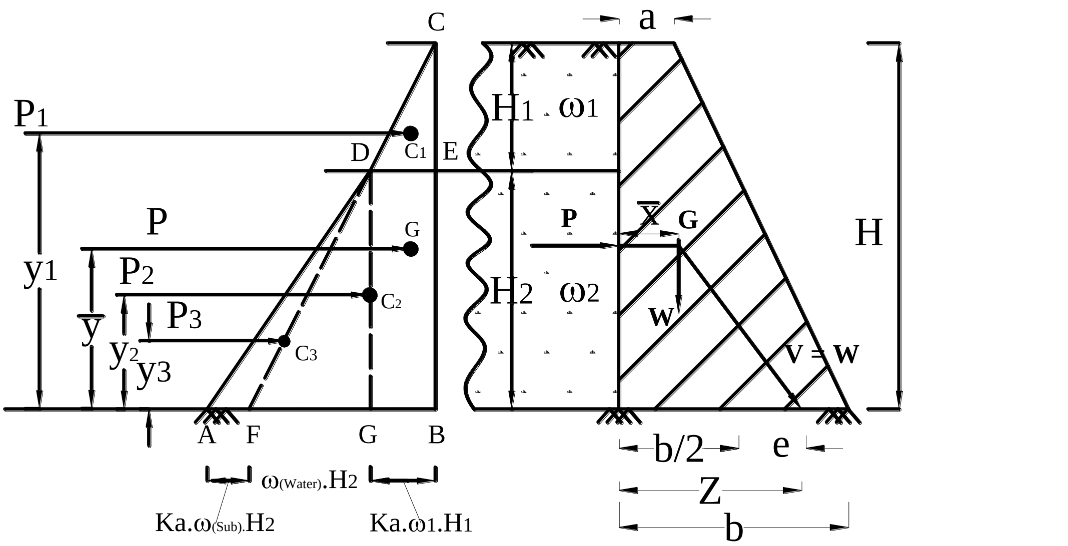
i. Weight of wall/m run (W)

$$W = \gamma\ (\frac{(a+b)}{2} x H = \frac{23(2+4)}{2} \text{ x } 6 = 414 \text{ kN}$$

This will act at a distance $\bar{x}$ from Vertical face

$$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} = \frac{2^2+(2x4)+4^2}{3\,(2+4)} = 1.55\text{m}$$

Draw pressure diagram as shown in fig.



$$K_a = \frac{(1-\sin\phi)}{(1+\sin\phi)} = \frac{1-\sin 30}{1+\sin 30} = \frac{1}{3} = 0.333$$

Pressure Diagram **Section**

Intensity of pressure due to top soil to a depth of (H_1) = $K_a\, x\, \omega_1\, H_1$

= (0.333 x 16 x 4.5) = 24 kN/m^2

Intensity of pressure due to water to a depth of (H_2) = $\omega_{water}\, x\, H_2$

= 10 x 1.5 = 15 kN/m^2

Intensity of pressure due to submerged soil = $K_a\, x\, \omega_{(sub)}\, x\, H_2$

= 0.33 x 9 x 1.5 = 4.5 kN/m^2

i. Total horizontal pressure/m run (P)

P = $P_1 + P_2 + P_3$

P_1 = Pressure due to top soil to a depth 'H_1'

= Area of Δ^{le} CDE = $\frac{1}{2}$ x (K_a x ω_1 H_1) H_1

P_1 = $\left(\frac{1}{2} x 24 x 4.5\right)$ = 54 kN

This will act at y_1 distance from base.

y_1 = $\left(H_2 + \frac{H_1}{3}\right) = 1.5 + \left(\frac{4.5}{3}\right)$ = 3.0m

P_2 = Pressure due to top soil to a depth H_2

= Area of rectangle BCEF

= (Ka ω_1 H_1) x H_2 = 24 x 1.5 = 36 kN

This will act at a depth y_2 from base

y_2 = $\frac{H_2}{2} = \frac{4.5}{2}$ = 2.25m

P_3 = Pressure due to water and due to submerged soil

= Area of triangle AFE

= $\frac{1}{2}$ x (K_a x $\omega_{(sub)}$ x H_2 + ω_{water} x H_2)

= $\frac{1}{2}$ [(0.333 x 9 x 1.5) + (10 x 1.5)] x 1.5

P_3 = $\frac{1}{2}$ (4.5 + 15) x 1.5 = 14.625 kN

This will act at 'y_3' distance from base

y_3 = $\frac{H_2}{3} = \frac{1.5}{3}$ = 0.5m

$\therefore$ P = P_1 + P_2 + P_3 = (54 + 36 + 14.625) = 104.625 kN

This total horizontal pressure will act at $\bar{y}$ from base.

Taking moment about bottom

P x $\bar{y}$ = (P_1 y_1 + P_2 y_2 + P_3 y_3)

104.625 $\bar{y}$ = (54 x 3) + (36 x 2.25) + (14.625 x 0.5) = 250.31

$\bar{y}$ = $\left(\frac{250.31}{104.62}\right)$ = 2.39m

iii. Position of resultant thrust (Z)

Z = $\bar{x}$ + $\frac{P}{W}$ x $\bar{y}$

Z = 1.55 + $\left(\frac{104.625}{414}\right)$ x 2.39 = 2.15m

Eccentricity e = (Z - $\frac{b}{2}$) = (2.15 - $\frac{4}{2}$)

e = 0.15m

iv. Stresses at the base

σ = $\frac{W}{b}$ $\left(1 \pm \frac{6e}{b}\right)$

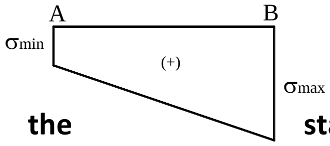
σ_{max} = $\frac{W}{b}$ $\left(1 + \frac{6e}{b}\right)$ at base

$$\sigma_{max} = \frac{414}{4}\left(1+\frac{6\,x\,0.15}{4}\right) = 126.78 \text{ kN/m}^2 \text{ (Comp.)}$$

$$\sigma_{min} = \frac{W}{b}\left(1-\frac{6e}{b}\right) \text{ at heel}$$

$$\sigma_{min} = \frac{414}{4}\left(1-\frac{6\,x\,0.15}{4}\right) = 80.21 \text{ kN/m}^2 \text{ (Comp.)}$$

Stress diagram



Check the stability

i. To avoid tension

e ≤ $\frac{b}{6}$ = $\frac{b}{6} = \frac{4}{6}$ = 0.6 mm

e = 0.15 < $\frac{b}{6}$ **Hence no tension at base.**

ii. To avoid overturning

$$\text{F.S} = \frac{W\,(b-\bar{x})}{P\,x\,\bar{h}} \geq 1.0 \qquad = \frac{414(4-1.55)}{104.625\,x\,2.39} = 4.0 > 1.0$$

Hence safe against overturning.

iii. To avoid sliding

$$\text{F.S} = \frac{\mu\,x\,W}{P} = \frac{0.6\,x\,414}{104.625} = 2.37 > 1.00$$

Hence safe.

Solved Problems:

Problem:1

A trapezoidal masonry retaining wall 1.2m wide at top, 3.6m wide at base is 6m high. The vertical face retains earth up to the top with an angle of repose of soil as 30^0. Take weight of masonry as 23 kN/m^3 and that of earth as 16KN/m^3. Check the sliding of retaining wall for overturning and sliding. If μ=0.60 and F.O.S= 1.5

Given data:

Top width, a=1.2m

Bottom width, b=3.6m

Height, H=6m

Angle of repose of soil ϕ= 30^0

Weight of masonry, γ_m =23 KN/m^3

Weight of soil, γ = 16 KN/m^3

To Find :

(i) $\sigma = \frac{V}{b}\left(1 \pm \frac{6e}{b}\right)$

(ii) Check the stability of retaining wall for overturning and sliding.

Solution:

(i) weight of wall per meter run (W)

$W = \gamma_m \left(\frac{a+b}{2}\right) H$

$=23 \left(\frac{1.2+3.6}{2}\right) x6$

$w = 331.2 \text{ kN}$

This will act as $\bar{x}$ distance from vertical face

$\bar{X} = \frac{a^2+ab+b^2}{3(a+b)}$

$= \frac{1.2^2+(1.2x3.6)+3.6^2}{3(1.2+3.6)} = 1.3 \text{ m}$

(ii) rankine's earth pressure per meter run (P)

$P= \frac{K_a\ \gamma.H^2}{2}$

$k_a = \frac{1-\sin\Phi}{1+\sin\Phi}$

$= \frac{1-\sin 30^0}{1+\sin 30^0} = \left(\frac{1-o.5}{1+0.5} \right)$

$k_a = \left(\frac{0.5}{1.5}\right) = 0.33$

$P = 0.33x\left(\frac{16 \text{ x } 6^2}{2}\right)$

P = 95.04 kN.

It will act at a distance $\bar{h}$ from base

$\bar{h} = \frac{H}{3} = \frac{6}{3} = 2m$

(iii) resultant thrust (R) per meter run

$R = \sqrt{W^2 + P^2} = \sqrt{331.2^2 + 95.04^2} = 344.56 \text{ kN}$

(iv) position of resultant thrust (R)

$z = \bar{x} + \frac{P}{W} + \bar{h}$

$= 1.3 + \left(\frac{95.04}{331.2} x2\right) = 1.87 \text{ m}$

z =1.87 m

(v) eccentricity

$e = z - \frac{b}{2}$

$= 1.87 - \frac{3.6}{2} = 0.07m$

e = 0.07 m

(vi) stress at base (σ)

$\sigma = \frac{w}{b} \left(1 \pm \frac{6e}{b} \right)$

$\sigma_{max} = \frac{w}{b}\left(1+\frac{6e}{b}\right)$

$= \frac{331.2}{3.6} \left(1 + \frac{6(0.07)}{3.6} \right)$

$=102.73 \text{ kN/m}^2$

$\sigma_{max}= 102.73 \text{ kN/m}^2$ (compression)

$\sigma_{min} = \frac{w}{b}\left(1-\frac{6e}{b}\right)$

$$= \frac{331.2}{3.6}\left(1-\frac{6(0.07)}{3.6}\right) = 81.267\ kN/m^2$$

σ_{min}= 81.267 kN/m^2 (compression)

Check the stability of wall

To avoid over turning

$$F.S = \frac{W\,(b-\overline{x})}{P\bar{h}}$$

$$= \frac{331.2\ (3.6-1.3)}{95.04\ x\ 2} = 4 > 1.0 \text{ (Hence safe)}$$

Hence safe against overturning

To avoid sliding

$$F.S = \frac{\mu W}{P} = \geq 1.0$$

$$F.S(sliding) = \frac{0.6\ x\ 331.2}{95.04} = 2.07$$

$2.09 \geq 1.0$

Hence safe against sliding

Problem:2

A retaining wall 7.5m high a vertical face supports loose earth at a surcharged of 20^0 to the horizontal. If the earth has of repose of 35^0 and has an angle. Specific weight of 16KN/m^3. Calculate the earth pressure per meter length of wall by rankine's formula. Also Calculate the horizontal and vertical components of the above earth pressure

Given data:

Height of wall h=7.5 m

Surcharge angle $\alpha = 20^0$

Angle of repose $\Phi = 35^0$

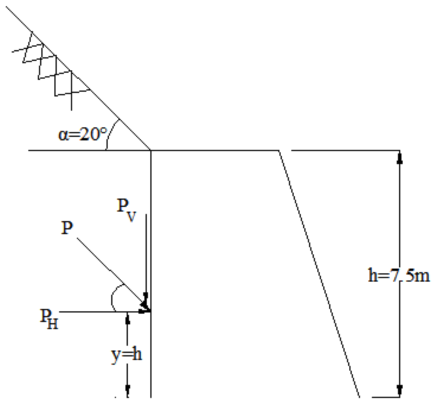
Specific weight of earth γ_e =16kN/m^3

To Find

(i) earth pressure =?

(ii) Horizontal pressure = ?

(iii) Vertical pressure=?



Solution:

Co -efficient of active earth pressure

$$k_a = \cos\alpha\ x \frac{\cos\alpha - \sqrt{\cos^2\alpha - \cos^2\phi}}{\cos\alpha + \sqrt{\cos^2\alpha - \cos^2\phi}}$$

$$= \cos 20^0 x \frac{\cos 20^0 - \sqrt{\cos^2 20^0 - \cos^2 35^0}}{\cos 20^0 + \sqrt{\cos^2 20^0 - \cos^2 35^0}}$$

$$= 0.9397\ x \frac{0.9397 - \sqrt{0.883 - 0.67}}{0.9397 + \sqrt{0.883 - 0.67}} = 0.57$$

$=0.32$

Earth pressure per meter length oh wall

$$p = \frac{k_a \cdot \gamma_e \cdot h^2}{2}$$

$$= \frac{0.32X16X7.5^2}{2}$$

P =144kN

Horizontal component,

$P_H = P\cos \propto$

$= 144\ x\ \cos 20^0$

P_H = 135.32 kN

Vertical pressure component,

$P_V = P\sin \propto$

$= 44\ x\ \sin 20^0$

P_V= 49.25 kN

Result:

(i) Earth pressure per meter length of wall P=256.5 kN

(ii) Horizontal component, P_H = 241.03 kN

(iii) Vertical component, P_V = 87.7 kN

Problem:3

A masonry retaining wall of 1m wide at top and 3m wide at bottom retains earth on the vertical face up to the top. Calculate the maximum height of the wall to ensure safety against sliding. If the factor of safety against sliding is 1.5 co-efficient of friction is 0.6. specific weight of masonry is 22KN/m^3 and specific weight of soil is 18KN/m^3 angle of repose 30^0

Given data:

Top width of wall, a=1m

Bottom width of wall, b= 3m

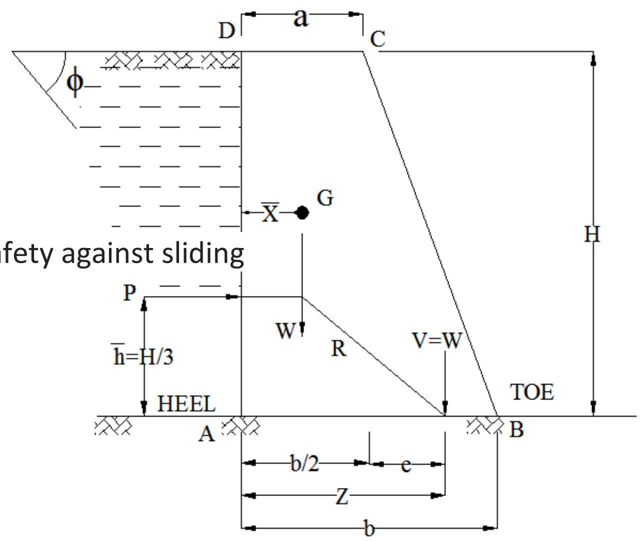
F.O.S against sliding = 1.5

Specific weight of masonry γ_m = 22 KN/m^3

Co – efficient of friction, μ = 0.6

Angle of repose, Φ = 30^0

Specific weight of soil γ =18 kN/m^3



To Find:

Maximum height of the wall to ensure safety against sliding

Solution:

Co-efficient of action earth pressure

$$k_a = \frac{1-\sin\phi}{1+\sin\phi}$$

$$= \frac{1-\sin 30^0}{1+\sin 30^0} = \left(\frac{1-0.5}{1+0.5}\right)$$

$$=\left(\frac{0.5}{1.5}\right) = 0.333$$

Lateral earth pressure

$$P = \frac{k_a.\gamma.h^2}{2}$$

$$= 0.333x\left(\frac{18.h^2}{2}\right)$$

P = 3h2

Weight of retaining wall

$$W = \frac{\gamma_{m.h}}{2}(a+b)$$

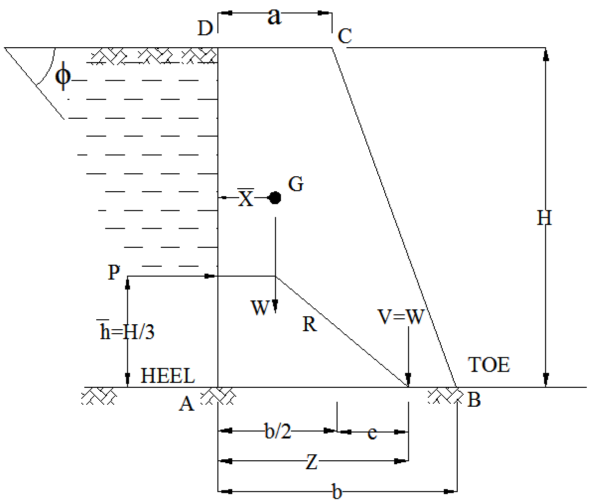
$$= \frac{22x\ h}{2}(1+3)$$

W= 44h

To avoid sliding $F.O.S = \frac{\mu W}{P}$

$$1.5 = \frac{0.6 \text{ x } 44h}{3h^2}$$

h= 5.87m



Problem:4

A retaining trapezoidal in section 10m high,1m wide at top, 3.2 m wide at bottom with a vertical face retains earth level with the top of the wall. If the weight of the masonry is 24 kN/m^3 and that of earth is 18 kN/m^3 with an angle of repose of 40^0. Calculate the maximum and minimum stresses at the base.

Given data:

Height of wall, h =10m

Top width, a =1m

Bottom width, b =3.2m

Specific weight of earth, γ_e= 18 kN/m^3

Specific weight of masonry, γ_m =24 kN/m^3

Angle of repose of soil, Φ =40^0

To Find:

Maximum and minimum stresses at the base.

i.e., σ_{max} and σ_{min}

Co-efficient of active earth pressure

$k_a = \frac{1-\sin\phi}{1+\sin\phi}$

$= \frac{1-\sin 40^0}{1+\sin 40^0}$ = 0.217

Lateral earth pressure

P = $\frac{k_a.\gamma.h^2}{2}$

= 0.217x($\frac{18x10^2}{2}$)

P = 195.30 kN

Weight of retaining wall

W = $\frac{\gamma_{m.h}}{2}$ (a+b)

= $\frac{24x10}{2}$(1+3.2)

W= 504 kN

$\overline{X} = \frac{a^2+ab+b^2}{3(a+b)}$

$= \frac{1^2+(1x3.2)+3.2^2}{3(1+3.2)}$

=1.146 m

Z = $\overline{x} + (\frac{P}{W} \text{ x } \frac{h}{3})$

= 1.146+ $(\frac{195.30}{504} \text{x} \frac{10}{3}) = 2.44\text{m}$

Z = 2.44m

eccentricity (e),

e = z - $\frac{b}{2}$

= 2.44 – $\frac{3.2}{2}$ = 0.84 m

e = 0.84 m

stress at base (σ)

σ = $\frac{w}{b}$ $(1 \pm \frac{6e}{b})$

$\sigma_{max} = \frac{w}{b}(1+\frac{6e}{b})$

= $\frac{504}{3.2}$ $(1 + \frac{6(0.8)}{3.2})$ =393.75 kN/m^2

σ_{max}= 393.75 kN/m^2 (compression)

$\sigma_{min} = \frac{w}{b}(1-\frac{6e}{b})$

= $\frac{504}{3.2}$ $(1 - \frac{6(0.8)}{3.2})$ = - 78.75 kN/m^2

σ_{min}= 78.75 kN/m^2 (tension)

Result:

σ_{max}= 393.75 kN/m^2 (compression)

σ_{min}= 78.75 kN/m^2 (tension)

Problem:5

A retaining wall trapezoidal in section is 8m high. 1m wide at top and 3m wide at the bottom with a vertical earth face retains earth level with the top of the wall. If the weight of masonry is 24 kN/m^3 and that of the earth is 18 kN/m^3 with an angle of repose of 30^0. Calculate the maximum and minimum stress at base.

Give data:

Height, h =8m

Top width, a =1m

Bottom width, b =3m

Weight of masonry, γ_m= 24 KN/m^3

Weight of earth , γ_e= 18 KN/m^3

Angle of repose Φ =30^0

To Find:

check the stability of wall

Solution:

Co-efficient of action earth pressure

$$k_a = \frac{1-\sin\phi}{1+\sin\phi}$$

$$= \frac{1-\sin 30^0}{1+\sin 30^0} = 0.333$$

k_a = 0.333

Lateral earth pressure = $\frac{k_a \cdot \gamma_e \cdot h^2}{2}$

$$= \frac{0.333 \text{ x } 18 \text{ x } 8^2}{2}$$

P = 191.808 kN

Weight of retaining wall

$$W = \frac{\gamma_{m.h}}{2}(a+b)$$

$$= \frac{24\text{x } 8}{2}(1+3)$$

W= 384 kN

$$\overline{X} = \frac{a^2+ab+b^2}{3(a+b)}$$

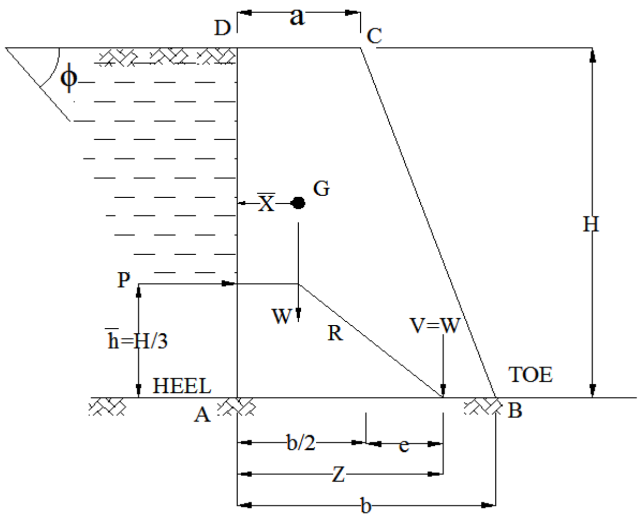
$$= \frac{1^2+(1\text{x}3)+3^2}{3(1+3)}$$

$\overline{X}$ =1.08 m

$$Z=\overline{x} + \frac{P}{W}\text{x}\frac{h}{3}$$

$$= 1.08+ (\frac{191.808}{384}\text{x}\frac{8}{3}) = 2.41\text{m}$$

Z =2.41 m



eccentricity

$$e = z - \frac{b}{2}$$

$$= 2.41 - \frac{3}{2} = 0.91\text{m}$$

e = 0.91 m

check for stability

(i) Check for tension at the base

e = 0.91m

$\frac{b}{6} = \frac{3}{6} = 0.5m$

e $> \frac{b}{6}$, the retaining wall is not safe against tension.

(ii)Check the overturning

Overturning moment = P x $\frac{h}{3}$

= 191.808 x $\frac{8}{3}$

=511.48 kN.m

Stabilizing moment = W (b - $\overline{x}$)

= 384 (3 – 1.08)

=737.28 kN.m

Factor of safety = $\frac{\text{stabilizing moment}}{\text{overturnig moment}}$

=$\frac{737.28}{511.48}$ = 1.44 < 1.5

The retaining wall is not safe

(iii) Check for sliding

Force causing sliding P= 191.808 KN

Force resisting sliding =µ.W

=0.6 x 384

= 230.4 kN

Force of safety = $\frac{\mu.W}{P}$

= $\frac{230.4}{384}$ = 0.6 < 1.5

The retaining wall is not safe against sliding

Problem:6

A retaining wall 2m wide at top, 4m wide at the base and 6m high retains earth to its full height on the vertical face. There is a road on the top of retaining earth which transmits uniform surcharged load of 60 KN/m^2. Take weight of masonry and earth are 24KN/m^3 and 18kN/m^3. An angle of earth as 40^0. Calculate the magnitude and the position of line of action of horizontal earth pressure per meter length of wall.

To Find :

- **(i) Horizontal pressure P=?**
- **(ii) Position of horizontal pressure $\bar{y}$ =?**

Solution :

(i) Co-efficient of active earth pressure (k_a)

$$k_a = \frac{1-\sin\phi}{1+\sin\phi} = \frac{1-\sin 40^0}{1+\sin 40^0} = 0.217$$

$$k_a = 0.217$$

Given data:

Top width, a=2m

Bottom width, b= 4m

Height, H= 6m

Uniform surcharged load,q= 60 kN/m^2

Angle of repose of soil, $\Phi=40^0$

Specific weight of earth, γ_e= 18 kN/m^3

Specific weight of masonry,γ_m= 24 kN/m^3

(ii) Pressure diagram

Intensity of pressure due to surcharged load at top = k_ax q

Intensity of pressure due to surcharged load at bottom =k_ax q

Draw pressure diagram of rectangle BCDB as shown in figure.

Intensity of pressure due to earth at top (h=0) =$k_a.h = 0$

Intensity of pressure due to earth at bottom (h=H) = $k_a.H$

Draw pressure diagram of triangle AED as shown in figure.

(iii) Horizontal earth pressure per meter run (P)

P=P_1+P_2

P_1= pressure due to surcharged load

P_2= area of rectangle BCDE

P_1= 0.218x60x6=78.48kN

This will act at y_1 distance from base

$Y_1= \frac{H}{2} = \frac{6}{2} = 3m$

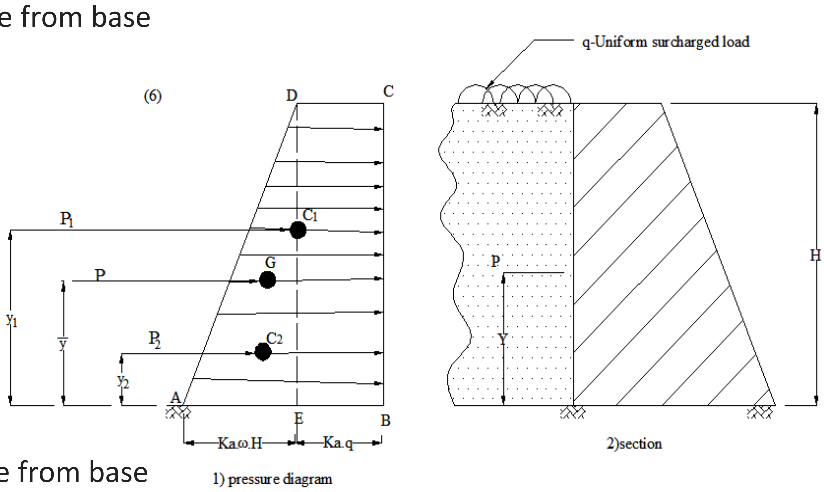
P_2= pressure due to retained earth

P_2=area of triangle AED

$P_2 = \frac{1}{2}x\ (k_a x\ \gamma_m x\ H)xH$

$P_2 = \frac{1}{2}$(0.217 x 18 x 6) x 6

P_2 =70.308 kN



This will act at y_2 distance from base

$Y_2= \frac{H}{3} = \frac{6}{3}$ =2m

Total horizontal pressure P =P_1+P_2

= 78.12+70.308=148.428 kN

This pressure will act at $\bar{y}$ distance from bottom

(iv) **Position of horizontal earth pressure (from base $\bar{y}$)**

$P x \bar{y} = P_1Y_1 + P_2Y_2$

$\bar{y} = \frac{1}{P}(P_1Y_1 + P_2Y_2)$

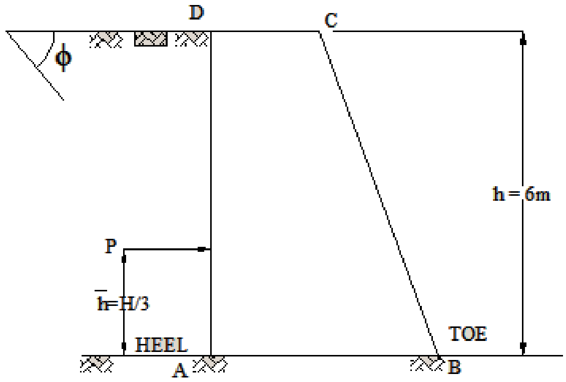
$= \frac{1}{148.428}(78.12x3+70.308x2)$

$\bar{y} = 2.56$ m from base

Result:

P=148.428kN

$\bar{y}$= 2.56m



Problem: 7

A retaining wall, 6m high with a smooth vertical back, retains earth level with the top of the wall. Determine the magnitude and line of action of the horizontal thrust per meter length of wall. The weight of sand is 20 kN/m^3 and its angle of repose is 40^0.

Given data

Height, h = 6m

Unit weight of sand, γ = 20 kN/m^3

Angle of repose , Φ = 40^0

To Find:

Horizontal thrust 'P' =?

Line of action of P,y =?

Solution

Co- efficient of active earth pressure,

$K_a = \frac{1-\sin\phi}{1+\sin\phi}$

$= \frac{1-\sin 40^0}{1+\sin 40^0} = 0.2174$

Horizontal thrust per meter length of wall,

$P = \frac{k_a.\gamma.h^2}{2}$

$= 0.2174 \times \frac{20x6^2}{2}$

= 78.264 kN

Line of action $y = \frac{h}{3} = \frac{6}{3} = 2$ m

Result

Horizontal thrust P = 78.264 kN acting of 2m above the base.

Problem: 8

A retaining wall of 5m high has a vertical earth face retains earth level with the top of the wall. In the top 3m, the weight of retained material is 20 kN/m^3 while below this level it is 24kN/m^3. The angle of repose of the two retained material is 30^0. The wall also carries an uniform surcharge load of 18kN/m^2 at the top. Calculate the magnitude and position of lateral thrust on the wall.

Given data

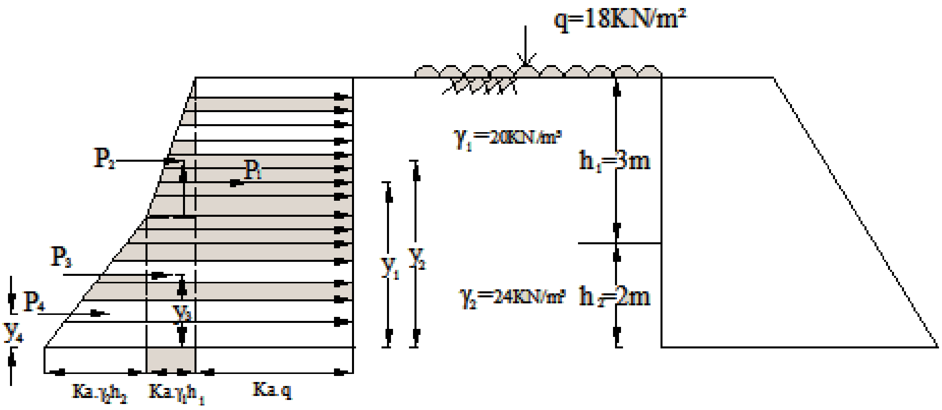
Height of wall, h = 5m

Sp. Weight of soil for the top 3m γ_1 = 20 kN/m^3

Sp. weight of soil for the bottom 2m γ_2 = 24kN/m^3

Surcharge load at top q = 18 kN/m^2

Angle of repose, Φ = 30^0



To Find:

Horizontal thrust P = ?

Line of action of horizontal thrust, y =?

Solution

Co- efficient of action earth pressure,

$k_a = \frac{1-\sin\phi}{1+\sin\phi}$

$= \frac{1-\sin 30^0}{1+\sin 30^0} = 0.333$

P_1 = pressure due to surcharge load 'q'

$= k_a.q.h = 0.333 \times 18 \times 5 = 29.97$ kN

$Y_1 = \frac{h}{2} = \frac{5}{2} = 2.5$ m above the base.

Pressure due to the top soil of γ_1 = 20 kN/m^3

$P_2 = \frac{k_a.\gamma_1.h_1^2}{2} = 0.333 \times \frac{20 \times 3^2}{2} = 29.97$ kN

$Y_2 = 2 + \frac{1}{3} \times 3 = 3$ m above the base

$P_3 = k_a.\gamma_1.h_1.h_2 = 0.333 \times 20 \times 3 \times 2 = 39.96$ kN

$Y_3 = \frac{2}{2} = 1$m above the base

Pressure due to the bottom soil of γ_2 = 24 kN/m^3

$P_4 \;=\; \frac{k_a.\gamma_2.h_2^2}{2} = 0.333 \text{ x } \frac{24 \text{ x } 2^2}{2} = \quad 15.984 \text{ kN}$

$Y_4 \;= \frac{1}{3} \text{ x } 2 \;=\; 0.667$ m above the base

Total horizontal earth pressure,

$P = P_1+P_2+P_3+P_4$

= 29.97 + 29.97 + 39.96 + 15.984

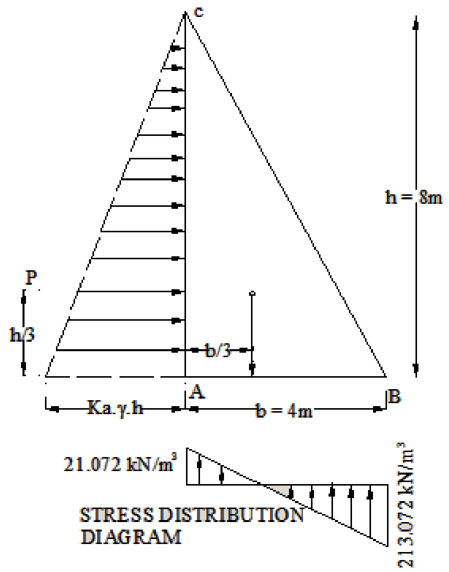
= 115.884 kN

Line of acting of P ,

$Y = \frac{P_1.Y_1+P_2Y_2+P_3Y_3+P_4P_4}{P_1+P_2+P_3+P_4}$

$= \frac{29.97 \text{ x } 2.5+29.97 \text{ x } 3+39.96 \text{ x } 1+15.984 \text{ x } 0.0667}{115.884}$

= 1.776 m above the base



Result:

Total horizontal earth pressure, P = 115.884 kN acting at y = 1.776 m the base.

Problem: 9

A retaining wall triangular in section is 8m high and 4m wide at the base, with a vertical face retaining earth level with the top of the wall. Draw curves of variation of (1) intensity of earth pressure on the vertical face. (2) the nominal stress intensity on the base, if the specific weight of earth is 20 kN/m^3 with an angle of repose of 30^0 and the specific weight of masonry is 24 kN/m^3.

Given data:

Height of wall,   h = 8m

Bottom of width, b = 4m

Specific weight of earth, γ_e = 20 kN/m^3

Specific weight of masonry, γ_m = 24 kN/m^3

Angle of repose of soil, Φ = 30^0

To Find:

(1) Intensity earth pressure on the vertical face.

(2) Nominal stresses at the base

Solution

Co- efficient of action earth pressure,

$k_a = \frac{1-\sin\phi}{1+\sin\phi}$

$= \frac{1-\sin 30^0}{1+\sin 30^0} = 0.333$

(1)Intensity of earth pressure on the vertical face:

Intensity of earth pressure at top   = 0

Intensity of earth pressure at the base = $k_a.\gamma_e$. h

= 0.333 x 20 x 8

= 53.28 kN/m^3

Lateral earth pressure,

$P = \frac{k_a.\gamma.h^2}{2}$

$= \quad 0.333 \times \frac{20x8^2}{2}$

= 213.12 kN

Weight of retaining wall,

$W = \frac{\gamma_m.h}{2}\ (a+b)$

$= \frac{24\ x\ 8}{2}\ (0+4)$ = 384 kN

$\bar{x} = \frac{b}{3} = \frac{4}{3}$ = 1.333 m

$Z = \bar{x} + \frac{P}{W} x \frac{h}{3}$

$= 1.333 + \frac{213.12\ x\ 8}{384\ x\ 3}$

= 2.813 m

$e = z - \frac{b}{2} = 2.813 - \frac{4}{2} = 0.813$ m

2) Nominal stresses at the base:

$\sigma_{max} = \frac{W}{b}\ (\ 1 + \frac{6e}{b}\ )$

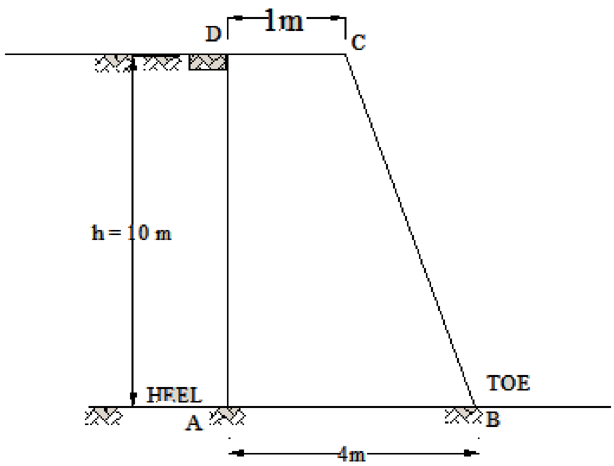
$= \frac{384}{4}\ (\ 1 + \frac{6x\ 0.813}{4}\ )$

= 213.072 kN/m^2 (comp)

$\sigma_{min} = \frac{W}{b}\ (\ 1 - \frac{6e}{b}\ )$

$= \frac{384}{4}\ (\ 1 - \frac{6x\ 0.813}{4}\ )$

= - 21.072 kN/m^2 (tension)



Result :

1. Intensity of earth pressure at the base = 53.28 kN/m^2

2.Nominal stress at the base,

σ_{max} = 213.072 kN/m^2 (comp)

σ_{min} = -21.072 kN/m^2 (tension)

Problem: 10

A masonry retaining wall is 10 m high and has a vertical face on the earth side. The top width of retaining wall is 1m and bottom width is 4m. The angle of repose is 30^0. Weight of earth is 18 kN/m^3 andweight of masonry is 24 kN/m^3. Check the stability of the retaining wall with respect to No tension state, (ii)over turning and (iii) sliding if μ = 0.6 and F.OS = 1.5.

Given data:

Height of wall h = 10m

Top width a = 1m

Bottom width b = 4m

Angle of repose $\Phi = 30^0$

Specific weight of earth $\gamma_e = 18\ kN/m^3$

Specific weight of masonry $\gamma_m = 24\ kN/m^3$

Co- efficient of friction, $\mu = 0.6$

Factor of safety F.O.S = 1.5

To Find:

Check the stability of wall

Solution

Co- efficient of active earth pressure

$k_a = \frac{1-\sin\phi}{1+\sin\phi}$

$= \frac{1-\sin 30^0}{1+\sin 30^0} = 0.333$

Lateral earth pressure

$P = \frac{k_a.\gamma.h^2}{2}$

$= 0.333 \times \frac{18x10^2}{2}$

= 299.97 kN

Weight of retaining wall,

$W = \frac{\gamma_m.h}{2}\ (a+b)$

$= \frac{24 \times 10}{2}\ (1+4) = 600$ kN

$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)}$

$= \frac{1^2+(1 \times 4)+4^2}{3(1+4)} = 1.4$m

$Z = \bar{x} + \frac{P}{W} \times \frac{h}{3}$

$= 1.4 + \frac{299.97}{600} \times \frac{10}{3}$

= 3.07 m

$e = z - \frac{b}{2} = 3.07 - \frac{4}{2} = 1.07$ m

Check for stability

(i)Check for tension at the base

e = 1.07 m

$\frac{b}{6} = \frac{4}{6} = 0.667$ m

$e > \frac{b}{6}$, the retaining wall is not safe against tension.

(ii) Check for over turning

Over turning moment = $P \times \frac{h}{3} = 299.97 \times \frac{10}{3} = 999.9$ kN.m

Stability moment = $W (b - \bar{x})$

= 600 (4 – 1.4) = 1560 kN.m

Factor of safety = $\frac{\text{stabilizing moment}}{\text{over turning moment}}$

$= \frac{1560}{999.9} = 1.56 > 1.5$

The retaining wall is safe against over turning

(iii)Check for sliding

For causing sliding = P = 299.97 kN

Force resisting sliding = μ . W

= 0.6 x 600 = 360 kN

Force of safety = $\frac{\mu . W}{P}$

= $\frac{360}{299.97}$ = 1.2 < 1.5

The retaining wall is not safe against sliding.

HIGHLIGHTS

Difference between active and passive earth pressure

Sl No	Active earth pressure	Passive earth pressure
1	The lateral pressure exerted by the retained earth on the retaining wall is known as active earth pressure.	The lateral pressure exerted by the retaining wall on the retained soil is known as passive earth pressure.
2	The wall tends to move away from the soil.	The retaining wall tends to move against the soil.
3	It is due to the expansion of soil.	It is due to the contraction of soil.
4	The angle made by the failure plane with horizontal, i= 45+$\frac{\phi}{2}$	The angle made by the failure plane with horizontal i=45 - $\frac{\phi}{2}$
5	The intensity of active earth pressure is given by P_a= k_a . γ . h. where h= height of soil retained and k_a= co- efficient active earth pressure. = $\frac{1-\sin\phi}{1+\sin\phi}$	The intensity of passive earth pressure is given by P_p= k_p . γ . h. where h= height of soil retained and k_a= co- efficient passive earth pressure. = $\frac{1+\sin\phi}{1-\sin\phi}$
6	The active earth pressure is the practical pressure which acts on the retaining wall.	The passive earth pressure is the theoretical pressure which rarely comes into play.

REVIEW QUESTIONS

Two mark questions

1. Define angle of repose of soil.
2. Write the Rankine's total earth pressure formula for the retaining wall retaining earth with level back fill.
3. Write the Rankine's total earth pressure formula for the retaining wall retaining earth with angular surcharge.
4. What is an active earth pressure?
5. What is passive earth pressure?
6. What are state of equilibrium of soil?
7. What is meant by plastic equilibrium of soil?
8. What is an elastic equilibrium of soil?
9. At which point of the base, the bearing pressure will be maximum in a retaining wall?
10. What will be the co-efficient of passive earth pressure of soil having angle of repose 30°?

Three marks

1. Explain state of equilibrium of soil.
2. Explain active and passive earth pressure in retaining wall.
3. Explain the angle of repose of soil.
4. Distinguish between active and passive earth pressure.
5. What are the assumption made in Rankine's theory of earth pressure?
6. Explain the Rankine's theory of earth pressure.
7. What are the forces acting on the retaining will to keep it in equilibrium? Draw the normal stress distribution diagram to avoid tension at the base.
8. State the conditions to check the stability of retaining wall.
9. Develop Rankine's total earth pressure formula for the retaining wall retaining earth with uniform surcharge.
10. Develop Rankine's total earth pressure formula for the retaining wall retaining earth with angular surcharge.

Ten marks

1. A retaining wall trapezoidal in section is 9m high, 2m wide at top and 3m wide at the bottom with a vertical earth face retaining earth level with the top of wall. If the weight of masonry is 24kN/m^3 and that of the earth is 18 kN/m^3 with an angle of repose of 30°. Calculate the maximum and minimum stress at the base.
2. A retaining wall 7.5m high with a vertical face supports loose earth at a surcharge of 20° to the horizontal, if the earth has an angle of repose of 35° and has a specific weight of 20 kN/m^3. Calculate the earth pressure per meter length of wall by Rankine's formula. Calculate the horizontal and vertical components of the above earth pressure.

3. A retaining wall 6m high with a smooth vertical back retains earth level with the top of the wall. Determine the magnitude and the line of action of the horizontal thrust per meter length of wall. The weight of sand is 20 kN/m^3 and its angle of repose 40°.
4. A trapezoidal masonry retaining wall 1m wide at top, 2m wide at the base is 7.5m high. It retains earth on its vertical face with the top of the wall. The angle of repose of soil is 30°. Take weight of masonry as 22 kN/m^3 and weight of earth as 18 kN/m^3. Check the stability of retaining wall, if the co-efficient of friction between masonry and soil as 0.60 and factor of safety as 1.50.
5. A retaining wall 2m wide at the top, 4m wide at the base and 6m high retains earth to its full height on the vertical face. There is road on the top of retained earth which transmits uniform surcharged load of 50 kN/m^2. Take weight of earth as 18 kN/m^3 and angle of repose of earth as 30°. Calculate the magnitude and the position of line of action of horizontal earth pressure per meter length of wall.
6. A trapezoidal masonry retaining wall 1.5m wide at the top, 5m wide at the base and 9m high with a vertical face retains earth level with the top of the wall at 2m below the top level, the foundation of structures transmits a uniform surcharged load of 120 $kN/m^{2.}$ Take weight of earth as 20 kN/m^3 and angle of repose of earth as 35°. Calculate the magnitude and the position of the earth pressure on the retaining wall.